Note on 1D - Bosonization Technique

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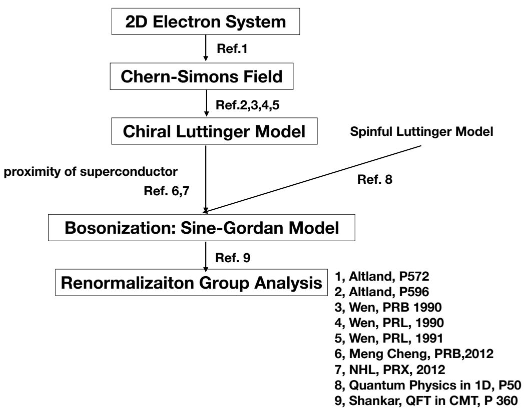
Abstract

Physics does not depend on the representation, but your life surely does.

In this short note, we review the basic bosonization method studying the interacting 1D system(spinless Luttinger Model). Then we review the fractional quantum hall effect(FQH) and spinful Luttinger Model and apply it to several state-of-art experiment. Here comes a figure to demonstrate the main references and road map.



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1 Luttinger Model: Interacting electron in 1D

We consider the nearly free electron in 1D, which is called Luttinger model. The Hamiltonian reads:

$$\boxed{H = \sum_k a_k^\dagger \left(\frac{k^2}{2m} - E_F \right) a_k + \frac{1}{2L} \sum_{k,k',q\neq 0} V\left(q\right) a_{k-q}^\dagger a_{k'+q}^\dagger a_{k'} a_k} \tag{1.1}$$

Where $a_k^\dagger$ denotes creation of a spinless quasi-electron(and you can recover the spin band then, or you can simply consider a fully spin-polarized band) with the momentum k, E_F is the chemical potential, L is the length of the system and $V(q) = \frac{4\pi}{q^2}$ denotes the interaction between different electron.

We first split the Hamiltonian into a non-interacting part and the interacting term, i.e $H = H_0 + V(q)$. The most particular feature that highlights the 1D system is that only density fluctuations are essential. One can picture this by the fact that optimizing the 1D fermions, electrons can merely push each other but in higher-dimension, electron can move around to reach the optimal state.

The 1D system's Fermi surface, only containing two isolated points $\{-k_F, k_F\}$, is also different from the higher-dimensional case.

We first try to reshape the free Hamiltonian, in order to obtain some insights about the problem.

1.1 Expanding the free Hamiltonian

It's natural to expand the system near Fermi surface as the collective excitations are merely related to the physics that happens near the Fermi surface.

By using the Taylor expansion:

$$\frac{k^2}{2m} = \frac{k_F^2}{2m} + \sigma v_F \left(k - k_F\right) + O\left(k^2\right) \tag{1.2}$$

Where $\sigma = \pm$ denote the right-moving fermion(+) and the left-moving fermion($-$).

The Hamiltonian H_0 can be approximated as:

$$\boxed{H = \sum_{k,\sigma} v_F \left(\sigma k - k\right) a_{\sigma,k}^\dagger a_{\sigma,k}} \tag{1.3}$$

Which can be immediately recognized as a Dirac Hamiltonian. It should be always remembered that the summation is taken when the linear approximation is validate, i.e, $|k - k_F| < \Lambda$, a cut-off is added to the summation. As the basic excitation of this model is particle-hole excitation, one wants to test if the particle-hole quasi-excitation have well-defined momentum and other quantum number, which can be achieved by writing down the spectrum of particle-hole:

$$E_{R,k}\left(q\right) = E_R\left(k+q\right) - E_R\left(k\right) = v_F\left(k+q\right) - v_F k = v_F q \tag{1.4}$$

Thus, the linear system have well-defined particle-hole excitations. So it's then natural to using the corresponding operator $\Sigma_k a_{\sigma,k+q}^\dagger a_{\sigma,k}$ to rewrite our Hamiltonian. This is a very important start point, from which we develop a whole method named 'bosonization'. One can also check directly that this operator is the Fourier transform of the density operator:

$$
\begin{aligned}
\rho_q &= \frac{1}{2\pi}\int dx e^{iqx}\rho\left(x\right) = \frac{1}{2\pi}\int dx e^{iqx} a_x^{\dagger} a_x \\
&= \left(\frac{1}{2\pi}\right)^3 \sum_{k,k'} \int dx e^{iqx} e^{ikx} a_k^{\dagger} e^{-ik'x} a_{k'} = \sum_k a_{k+q}^{\dagger} a_k
\end{aligned}
\tag{1.5}
$$

And naturally $\rho_q^{\dagger} = \rho_{-q}$ as the density operator is Hermitian.

1.2 Calculating the of the density operator

Before we go further to demonstrate the idea, we should be cautious to avoid infinities when treating the Fourier transformation of (i.e, a linear combination of) the density operator. In Dirac-like Hamiltonian, it is permitted that infinite occupation below the Fermi energy. To this end, we introduces so-called normal ordering tricks:

In a normal ordered product, the destruction operators are put on the right and creation operators on the left. For two operator A, B that are linear combinations of creation and destruction operator, normal ordering operation is equivalent to the original operator subtracting the average value of the operator in the vacuum, i.e:

$$
\boxed{: AB := AB - \langle 0|\, AB \,|0\rangle}
\tag{1.6}
$$

Then our goal is to figure out the commutation relations of this density operator, and take the linear superpositions of them to recover the Hamiltonian Eq(3). The fronter procedure can be accomplished by writing down:

$$
\begin{aligned}
\left[\rho_{\sigma,q}, \rho_{\sigma,-q'}\right] &= \sum_{k1,k2} \left[a_{\sigma,k1+q}^{\dagger} a_{\sigma,k1}, a_{\sigma,k2-q'}^{\dagger} a_{\sigma,k2}\right] \\
&= \sum_{k1k2} a_{k1+q}^{\dagger} a_{k2} \delta_{k1,k2-q'} - a_{k2-q'}^{\dagger} a_{k1} \delta_{k2,k1+q} \\
&= \sum_{k2} a_{\sigma,k2+p-p'}^{\dagger} a_{\sigma,k2} - a_{\sigma,k2-p'}^{\dagger} a_{\sigma,k2-p}
\end{aligned}
\tag{1.7}
$$

In the second line we omit the right-left index for simplicity. We may naively change the index of summation and conclude that the result is zero, but which turns out to be wrong:

$$
\sum_{k2} a_{\sigma,k2+p-p'}^{\dagger} a_{\sigma,k2} - \sum_{k2-p\to k2} a_{\sigma,k2+p-p'}^{\dagger} a_{k2} \overset{?}{=} 0
$$

The reason is that the Fourier-transformed bare density operators contain infinity number of occupied states and the equation becomes $\infty - \infty$ indefinite.(as an example, when k involves infinite modes, that $\sum_k b_k b_k^{\dagger} \overset{!}{=} \sum_k 1 + b_k^{\dagger} b_k = \infty + \sum_k b_k^{\dagger} b_k$) So we try to use the normal ordering tricks to evaluate this quantity because the matrix element of normal ordering operator is always finite. As a matter of fact:

$$[\rho_{\sigma,q'},\rho_{\sigma,-q'}] = \sum_{k2} :a^\dagger_{\sigma,k2+q-q'}a_{\sigma,k2}: - :a^\dagger_{\sigma,k2-q'}a_{\sigma,k2-q}: \\ + \sum_{k2} \langle 0|\, a^\dagger_{\sigma,k2+q-q'}a_{\sigma,k2}\,|0\rangle - \langle 0|\, a^\dagger_{\sigma,k2-q'}a_{\sigma,k2-q}\,|0\rangle \tag{1.8}$$

Then, the first subtraction can be safely evaluated as zero. Making use of the fact that $\langle 0|\, a^\dagger_{\sigma k}a_{\sigma k'}\,|0\rangle = \delta_{k,k'}$ we then get:

$$[\rho_{\sigma,q'},\rho_{\sigma,-q'}] = 0 + \delta_{q,q'}\sum_{k2}\langle 0|n_{\sigma,k2} - n_{\sigma,k2-q}\,|0\rangle \tag{1.9}$$

One may then naively to conclude that the second term equals zero because at a first glance

$$\sum_{k2}\langle 0|n_{\sigma,k2}\,|0\rangle \stackrel{?}{=} \sum_{k2}\langle 0|n_{\sigma,k2-q}\,|0\rangle$$

But this is not true because we actually have a cut-off in momentum! Since the shift $k \to k-q$ changes the cut-off, the result is exactly the number of states in the interval $[k, k+q]$. Due to the well defined properties of the density in 1-D system, the result is $\frac{|q|}{2\pi/L}$ and independent of the cut-off properties of Λ. We arrive at the conclusion:

$$\boxed{[\rho_{\sigma,q},\rho_{\sigma',-q'}] = -\delta_{\sigma,\sigma'}\delta_{q,q'}\frac{\sigma qL}{2\pi}} \tag{1.10}$$

The result is remarkable because of the bosonic properties of density operator arising from the large number of occupied states we consider. Thus we are able to recover the Hamiltonian we care about with bosonic density fluctuations.

1.3 Bosonization of Luttinger Model

We now construct the bosonic operator from the density operator. The method is direct to some extents, but it can also provide some insights.

Note that, in the RHS of the Eq 1.10, the commutator is momentum dependent. The natural way to remove this dependency is to partition the coefficients to each density operators, i,e , $\rho_{\sigma,q} \to \frac{\rho_{\sigma,q}}{(\frac{L|q|}{2\pi})^{1/2}}$.

Due to some sign considerations, one can finally (after some nontrivial trials) write down the bosonization solution for this problem:

$$b_q = n_q\rho_{+,q}, b^\dagger_q = n_q\rho_{+,-q} \\ b_{-q} = n_q\rho_{-,-q}, b^\dagger_{-q} = n_q\rho_{-,q} \tag{1.11}$$

Where $n_q = (\frac{2\pi}{L|q|})^{1/2}$ and $q > 0$.

Compactly and equivalently: $b^\dagger_q = n_q\sum_\sigma \Theta(\sigma q)\,\rho_{\sigma,-q}$, where Θ is the Heaviside theta function.

Having constructed the boson operator$(b, b^\dagger)$, we are now to find a representation. Before diving into the details, we can observe that a boson operator consists two fermion operator If we can use two boson operators to represent free Hamiltonian, it means that original two fermion operator should be mapped to quadratic fermion operators, e.g ,$a^\dagger a a^\dagger a = a^\dagger \left[\delta - a^\dagger a\right] a = \delta a^\dagger a - \left(a^\dagger a^\dagger a a\right)$(neglect the subscripts). This also indicates that, if it's possible to bosonize the free Hamiltonian, the interacting part is also quadratic and supposed to be bosonized.

The Hamiltonian of a system can be constructed in many forms but the evolution of the observable remains intact. As the Heisenberg equation indicate, the only important element is the commutation relation between the complete operator and the Hamiltonian. For example, we try to calculate the anti-commutation rules of $b_q(q > 0)$ and H_0:

$$\begin{aligned}
[b_q, H_0] &= n_q \sum_\sigma \left[\rho_{+,q}, v_F\left(\sigma k - k_F\right) a^\dagger_{\sigma,k} a_{\sigma,k}\right] \\
&= n_q \sum_{k,k_q} v_F\left(k - k_F\right)\left(a^\dagger_{+,k1-q} a_{+,k} \delta_{k1-k} - a^\dagger_{+,k} a_{+,k1} \delta_{k1-q-k}\right) \\
&= n_q \sum_k v_F q c^\dagger_{+,k-q} c_{+,k} = v_F q b_q
\end{aligned} \tag{1.12}$$

and similar relation holds when $q < 0$. [1]We can ultimately write down the transformed Hamiltonian:

$$H_0 \simeq const + \sum_{p \neq 0} v_F \left|p\right| b^\dagger_p b_p \tag{1.13}$$

And rewriting the interaction part is obvious:

$$\begin{aligned}
V &= \frac{1}{2\pi} \sum_{q>0} q \left(g_4 b_q b^\dagger_q + g_4 b^\dagger_{-q} b_{-q} + g_2 b_q b_{-q} + g_2 b^\dagger_{-q} b_q\right) \\
&= \frac{1}{2\pi} \sum_{q>0} q \begin{pmatrix} b_q & b^\dagger_{-q} \end{pmatrix} K_{ee} \begin{pmatrix} b^\dagger_q \\ b_{-q} \end{pmatrix} \\
K_{ee} &= \begin{pmatrix} g_4 & g_2 \\ g_2 & g_4 \end{pmatrix}
\end{aligned} \tag{1.14}$$

The interaction part is also quadric but contains non-particle number-conserving contributions. Together with free Hamiltonian, we can use Bogoliubov transformation to diagonalize the Hamiltonian which is a trivial task.

To diagonalize the Hamiltonian:

$$H = \sum_{q>0} q \Psi^\dagger_q K \Psi_q, \Psi_q = \begin{pmatrix} b^\dagger_q \\ b_{-q} \end{pmatrix} \tag{1.15}$$

We need to find a transformation to diagonalize the kernel matrix and also keep the generalized commutation relation intact. We find that:

$$C_{ij} = \left[\Psi_{q,i}, \Psi^\dagger_{q,j}\right] = \begin{pmatrix} -1 & \\ & 1 \end{pmatrix}_{ij} = \left(-\sigma_3\right)_{ij}$$

[1]Together this with Eq,1.10, we get U(1) Kac Moody algebra.

So the transformation matrix U should satisfy those condition:

$$\begin{aligned} H &= \sum_{q>0} q {\Psi'}_q^{\dagger} U^{\dagger} K U {\Psi'}_q \\ U^{\dagger} K U &= \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \\ U^{\dagger} \sigma_3 U &= \sigma_3 \end{aligned} \tag{1.16}$$

To this end, we find $\underbrace{\sigma_3 U^{\dagger} \sigma_3}_{U^{-1}} \sigma_3 K U = \sigma_3 K'$ which means$\sigma_3 K'$ contains the eigenvalues $\pm u$ on its diagonal. and $\sigma_3 K' = Diag(u, -u)$ means $K' = Diag(u, u)$. The eigenvalue of $\sigma_3 K$ can be readily computed as $u = \frac{\sqrt{(g_4+2\pi v_F)^2 - g_2^2}}{2\pi}$, we then arrive at the final result:

$$\boxed{H = u \sum_{q>0} q \Psi_q^{\dagger} {\Psi'}_q = u \sum_q |q| \, b_q^{\dagger} b_q} \tag{1.17}$$

The observables, particularly some thermodynamics, can be extracted from this Hamiltonian. The easiest calculation among them is the heat capacity:

$$\begin{aligned} C_V &= \frac{dE}{dT} = \frac{d}{dT} \sum_{p \neq 0} \varepsilon(p) f_B(\varepsilon(p)) \\ &= \beta^2 \sum_{p \neq 0} \varepsilon(p)^2 \frac{e^{\beta\varepsilon}}{\left(e^{\beta\varepsilon} - 1\right)^2} = \frac{T}{u} \left(\frac{L\pi}{3}\right) \end{aligned} \tag{1.18}$$

One can compare this result with free fermion gas case: $C_V = \frac{T}{v_F}(\frac{L\pi}{3})$

This kind of quasi excitations as the Bosonic operators describing the density fluctuations is termed charge density wave with the velocity u. In the non-interacting limit, where $g_2 = g_4 = 0$, the waves have a velocity v_F while $g_2 = 0, g_4 \neq 0$ will accelerate the wave. Intuitively, this can be interpreted as the nonzero g_4 is the fictitious interaction. But when $g_2 \neq 0$, the Coulomb interaction make it difficult for particles with opposite velocity move close to each other.

1.4 Further consideration

We are now at a stage that the Hamiltonian is quadratic combination of the boson p mode operator. To develop a corresponding field theory, we have a standard routine, which is exactly the inverse routine that we learned in standard QFT or Quantum Optics.

We then consider the transformed field operator using the same technique we developed before. The fermionic field operator is defined as the Fourier transformation of the mode annihilation operator:

$$a_\sigma(x) = \frac{1}{\sqrt{2\pi L}} \sum_k e^{ikx} a_{\sigma,k} \tag{1.19}$$

As the same spirit that the commutation relation can define the observables, we have :

$$[\rho_\sigma(q), a_\sigma(x)] = \frac{1}{\sqrt{2\pi L}} \sum_{k,k_1} e^{ik1x} \left[c^\dagger_{\sigma,k+q} c_{r,k}, c_{r,k1}\right] \\ = -e^{ipx} a_\sigma(x) \tag{1.20}$$

So, we can construct the field operator as $a_\sigma(x) \stackrel{!}{\simeq} e^{\sum_q e^{iqx}\rho_\sigma(q)\left(\frac{2\pi r}{qL}\right)}$. But it's not true, because the density operator never create or destroy any fermion but the field operator destroy a fermion in x. One wants to generate this as :

$$a_\sigma(x) = U_\sigma e^{\sum_q e^{iqx}\rho_\sigma(q)\left(\frac{2\pi r}{qL}\right)} \tag{1.21}$$

Where the U_σ should destroy a fermion of state σ and commute with the bosonic operator. These operators are known as 'Klein factor'. Using the conditions above, the dependence of the Klein factor on the boson operator can be derived. But in the first example, it's too tedious to do so, and few physics can be extracted by Klein factor, though this provides an exact and full transformation from fermionic representation to a bosonic one.

2 Bosonization with a field theory

We now want to develop a bosonic field theory about the Luttinger model. Put it in you heart that we even know the right answer to the theory because of the universality of the wave mechanism :

$$\mathcal{H} = \frac{\pi}{v_F}\Pi^2 + \frac{v_F}{2\pi}\partial_x\theta^2 \tag{2.1}$$

Thus, all we need to do is to reshape and map our original free system to this bosonic field theory. In other words, we want to recover the field Langragian density and Hamiltonian from the mode operator representation. To obtain an effective long range theory of the 1D interacting system, we have two strategies. The first one is to explictly construct the field theory from the mode operator and the action of the operator on Fock state. Another more phenomenological approach is based on consideration of the consistency and the symmetry acquirement. Although this method lacks quantitative evaluation of the results, we can gain some intuitions and even immediately obtain the right theory and finally make it rigorous.

2.1 Euclidean Fermionic Field Theory

We use the language of Euclidean fermionic field theory to represent this system:

$$S_0\left[\psi^\dagger, \psi\right] = \sum_\sigma \int dx d\tau \psi^\dagger_\sigma \left(-i\sigma v_F \partial_x + \partial_\tau\right)\psi_\sigma \\ S_{\text{int}}\left[\psi^\dagger, \psi\right] = \frac{1}{2}\sum_\sigma \int dx d\tau \left(g_2 \rho_\sigma \rho_{\sigma'} + g_4 \rho_\sigma \rho_\sigma\right) \tag{2.2}$$

As before, $\sigma = \pm$ and $+$ for right-moving fermions while $-$ for left-moving ones.

We will first show the phenomenological method. As we see, the 1D interacting system can hold density wave excitation, so the fermionic part should change the density of the system. As $N = \frac{L}{2\pi} 2k_F \to n_0 = \frac{k_F}{\pi}$, the first part of the bosonic operator is supposed to be $\frac{k_F}{\pi} + \rho(x)$. So we can write down the transformation:

$$b(x) = \left(\frac{k_F}{\pi} + \rho(x) \right)^{1/2} e^{i\phi(x)} \tag{2.3}$$

It can be easily shown that the transformation $(b, b^\dagger) \to (\rho, \phi)$ is canonical. For simplicity, we recover the total density of the system $\rho \to \rho + \frac{k_F}{\pi}$, we thus have:

$$\begin{aligned} 1 = \left[b, b^\dagger\right] = \left[\rho^{1/2} e^{i\lambda\phi}, e^{-i\lambda\phi} \rho^{1/2}\right] \\ = \rho - e^{-i\phi} \rho e^{i\phi} = \rho - e^{-i[\phi,]} \rho = \rho - \rho + i\left[\phi, \rho\right] - \frac{1}{2}\left[\phi, \left[\phi, \rho\right]\right] + ... \end{aligned} \tag{2.4}$$

The equality thus require $[\rho, \phi] = i$ whose complete and continuous version is:

$$[\rho(x), \phi(x')] = i\delta(x - x') \tag{2.5}$$

Based on our experience of Jordan-Wigner transformation, fermionic operator afford a representation of bosons: $a^\dagger(x) = e^{i\pi \int_{x'<x} dx' n(x')} b^\dagger(x), n(x) = b^\dagger(x) b(x)$ where $b(x)$ create a boson located at x and $\exp(i\pi \int_{x'<x} dx' n(x'))$ implements the fermionic statistics (What is fermion? Fermion is a hardcore boson in the string. This exponent operator serves as a string.) Thus, one can apply basic Jordan-Wigner transformation to construct fermionic representation. However, after simple trial you will find the solution is not corresponding to what we want, though right but useless. In this case, we can also implement this routine by define $\theta(x) = \pi \int_{-\infty}^{x} dx' \rho(x')$. Exponent of this will serve as a string operator. So what it the annihilator or creation operator? Note that the momentum conjugated to ρ is ϕ and the corresponding translation operator is $e^{i\phi(x)}$ creating a unity charge located at x. So the construction is naturally:

$$a^\dagger(x) = A \sum_\sigma e^{i\sigma\theta(x)/\pi} e^{i\phi(x)} \tag{2.6}$$

where A is introduced to regularize the RHS. And dividing it into the right-moving part and left-moving part:

$$a_\sigma^\dagger(x) = A e^{i\sigma\theta(x)/\pi} e^{i\phi(x)} \tag{2.7}$$

After shifting $\rho \to \rho + \frac{k_F}{\pi}$, and corresponding $\theta \to \theta + k_F x$, we get the final answer:

$$a^\dagger(x) = A \sum_\sigma e^{i\sigma\pi^{-1}[\theta(x) + k_F x]} e^{i\phi(x)} \tag{2.8}$$

And so does $a_\sigma^\dagger$ The commutation relation of our two main fields is

$$[\phi(x), \theta(x')] = i\pi\Theta(x' - x) \tag{2.9}$$

2.2 Bosonic Field in terms of bosonic mode

We've learned in basic quantum field theory that bosonic field operator can be represented by bosonic mode operator as:

$$\begin{aligned}\phi(x) &= \int \frac{dp}{2\pi\sqrt{2|p|}}\left[\phi_p e^{ipx} + \phi_p^\dagger e^{-ipx}\right] e^{-\frac{1}{2}\alpha|p|} \\ \Pi(x) &= \int \frac{dp}{2\pi\sqrt{2|p|}}\left[-i\phi_p e^{ipx} + i\phi_p^\dagger e^{-ipx}\right] e^{-\frac{1}{2}\alpha|p|}\end{aligned} \tag{2.10}$$

It's supposed to be emphasized that in bosonization theory, we have actually the same identity:

$$\begin{aligned}\phi(x) &= \lim_{L\to\infty}\sum_p \left(\frac{1}{2\pi L|p|}\right)^{1/2} Sign(p)\, e^{-\alpha|p|/2-ipx}\left(b_p^\dagger + b_{-p}\right) \\ \theta(x) &= \lim_{L\to\infty}\sum_p \left(\frac{1}{2\pi L|p|}\right)^{1/2} e^{-\alpha|p|/2-ipx}\left(b_p^\dagger - b_{-p}\right)\end{aligned} \tag{2.11}$$

One can easily check this by previous knowledge. After the construction of phenomenological field theory, we will come to this point later.

2.3 Construction of Phenomenological Field Theory

We now construct the action of this field theory. First, inspired by Eq. 2.7, we know that the axial and vectorial symmetry are required: shifting $(\phi,\theta) \to (\phi+\phi_v, \theta+\phi_a)$. For constant $\phi_{a,v}$, the action is intact. Thus the non-derivative terms are excluded.

Next, the density distortion $\rho = \partial_x\theta/\pi$ costs certain energy. Due to the screening effect, the coulomb interaction should be short range i.e local. Up to second order in derivatives, the action contain term like $(\partial_x\theta)^2$. To ensure rotational invariance in the 1 + 1-D space-time, the total action is given by $S_0[\theta] = \frac{c}{2}\int dx d\tau \left((\partial_\tau\theta)^2 + (\partial_x\theta)^2\right)$

Now our free theory have two undetermined parameter regularization coefficient A and the coupling constant c. As the free theory is integrable in fermion representation, we can compare the observables forecast by those two theories (e.g capacity). For example, we will calculate $C(x,\tau) = \left\langle \left(\psi_+^\dagger\psi_-\right)(x,\tau)\left(\psi_-^\dagger\psi_+\right)(0,0)\right\rangle$ for bosonic theory where we set $\tau > 0$. Corresponding quantity in fermionic theory is $C(x,\tau) = A^4\left\langle e^{2i\theta(x,\tau)}e^{-2i\theta(0,0)}\right\rangle_\theta$. By using Matsubara summation, we have :

$$\begin{aligned}G_\pm &= -\left.\frac{1}{\partial_{\tau'} \mp i\partial_{x'}}\right|_{(x,\tau;0,0)} \\ &= -\frac{T}{L}\sum_{p,\omega_n}\frac{1}{-i\omega_n \mp p}e^{-ipx-i\omega_n\tau} \\ &\simeq \frac{1}{2\pi}\frac{1}{\pm ix-\tau}\end{aligned} \tag{2.12}$$

We've used an integral to approximate the frequency sum and integrate over momentum. By wick's theorem, the correlation function is:

$$C = G_+ G_- = \frac{1}{4\pi^2} \frac{1}{x^2 + \tau^2}$$

For bosonic system. With the interaction $S[\theta] = \frac{L}{2cT} \Sigma_{q,n} (q^2 + \omega_n^2) |\theta_{q,n}|$ also by Matsubara summation, we have the green function for θ field:

$$\begin{aligned}
&K(x,\tau) = \left\langle \theta(x,\tau)\,\theta(0,0) - \theta^2(0,0) \right\rangle \\
&= \frac{cT}{L} \sum_{p,n} \frac{e^{ipx} + i\omega_n \tau}{\omega^2 + p^2} - 1 = \frac{cT}{2} \sum_n \frac{e^{-|\omega_n| x + i\omega_n \tau} - 1}{|\omega_n|} \\
&= \frac{c}{4\pi} \int_0^{a^{-1}} d\omega \frac{e^{-\omega(x - i\tau)} - 1}{\omega} + \frac{e^{-\omega(x + i\tau)} - 1}{\omega} \\
&\simeq -\frac{c}{4\pi} \ln \frac{x^2 + \tau^2}{a^2}
\end{aligned} \tag{2.13}$$

Where we add a cutoff a^{-1} and use stationary phase approxiamtion in large x and τ.

By using the fact $\left\langle e^{iU} \right\rangle = e^{-\frac{1}{2}\left\langle U^2 \right\rangle}$ Where U is the linear combination of ϕ and θ(See the prove in appendix), we have:

$$\begin{aligned}
&\left\langle e^{iU} \right\rangle = e^{-\frac{1}{2}\left\langle U^2 \right\rangle} \\
&C(x,\tau) = A^4 \left\langle e^{2i(\theta(x,\tau) - \theta(0,0))} \right\rangle \\
&= A^4 e^{-2\left\langle (\theta(x,\tau) - \theta(0,0))^2 \right\rangle} \\
&= A^4 e^{4K(x,\tau)} \\
&= A^4 \left[\frac{a^2}{x^2 + \tau^2} \right]^{\frac{c}{\pi}}
\end{aligned} \tag{2.14}$$

Thus, comparing 2.12 and 2.14 we have $A = \frac{1}{\sqrt{2\pi a}}$ and $c = \pi$ where a^{-1} is the cutoff.

We finally obtain the Lagrangian of the bosonic version:

$$\begin{aligned}
&\mathcal{L} = \frac{1}{2\pi} \left[(\partial_x \theta)^2 + (\partial_\tau \theta)^2 \right] \\
&\pi_\theta = \partial_{\partial_\tau \theta} \mathcal{L} = \frac{\partial_\tau \theta}{\pi}
\end{aligned} \tag{2.15}$$

Thus we have the commutation relation $[\theta(x), \pi(x')] = -i\delta(x - x')$. Comparing this with $[\phi(x), \theta(x')] = i\pi\Theta(x' - x)$, we thus have $\pi_\theta = \frac{\partial_x \phi}{\pi}$. Transforming the action from canonical variable to fundamental fields, we have:

$$\begin{aligned}
&S[\theta, \pi_\theta] = \frac{1}{2} \int dx d\tau \left(\frac{1}{\pi} (\partial_x \theta)^2 + \pi \pi_\theta^2 + 2i\pi_\theta \partial_\tau \theta \right) \\
&S[\theta, \phi] = \frac{1}{2\pi} \int dx d\tau \left((\partial_x \theta)^2 + (\partial_x \phi)^2 + 2i\partial_\tau \theta \partial_x \phi \right)
\end{aligned} \tag{2.16}$$

The corresponding conserved quantity of translation symmetry is

$$\begin{aligned} \rho_{total} &= \rho_+ + \rho_- = \frac{\partial_x \theta}{\pi} \\ j &= \rho_+ - \rho_- = -\frac{\partial_x \phi}{\pi} \end{aligned} \tag{2.17}$$

Thus we have the density operator in terms of field operators:

$$\rho_\pm = \frac{1}{2\pi}\left[\partial_x \theta \mp \partial_x \phi\right] \tag{2.18}$$

2.4 Interacting System

By inserting Eq 2.18 into the interaction vertex Eq (2.2), we have:

$$S_{\text{int}} = \frac{1}{4\pi^2}\int dx d\tau \left[(g_2 + g_4)(\partial_x \theta)^2 + (g_4 - g_2)(\partial_x \phi)^2\right] \tag{2.19}$$

Together with the free part, we arrive at the final expression:

$$S = \frac{1}{4\pi^2}\int dx d\tau \left[g^{-1} v (\partial_x \theta)^2 + g v (\partial_x \phi)^2 + 2i\partial_\tau \theta \partial_x \phi\right] \tag{2.20}$$

$$\begin{aligned} v &= \left[\left(v_F + \frac{g_4}{2\pi}\right)^2 - \left(\frac{g_2}{2\pi}\right)^2\right]^{1/2} \\ g &= \left[\frac{v_F + \frac{g_4}{2\pi} - \frac{g_2}{2\pi}}{v_F + \frac{g_4}{2\pi} + \frac{g_2}{2\pi}}\right]^{1/2} \end{aligned} \tag{2.21}$$

The velocity we introduce here is exactly the effective velocity we derived with mode operator and diagonalization in (1.17).

3 Functional Bosonization(Not finished yet)

3.1 Covariant Notation of Non-relativistic System

In this section, we use basic functional method to implement bosonization. In previous sections, we bosonize the theory by transforming the free part of the fermionic Hamiltonian. Now we approach the result with the idea of decoupling the quadric field by auxiliary fields[2].

We first rewrite the interaction part:

$$S_{\text{int}} = \frac{1}{2}\int d\tau dx \rho_i g_{ij} \rho_j \tag{3.1}$$

Where Einstein convention is assumed of $i, j = \{+, -\}$ for fields and $i, j = \{1, 2\}$ for matrix and $g_{11} = g_{22} = g_2, g_{12} = g_{21} = g_4$

By invoking the identity: $\exp\left[-\rho_m V_{mn} \rho_n\right] = \int \mathcal{D}\phi \left[-\frac{1}{4}\phi_m V^{-1}_{mn}\phi_n - i\phi_m \rho_m\right]$, we have:

[2]For more details, see e.g., [4]

$$S\left[\phi,\bar{\psi},\psi\right]=\int dxd\tau\left[\frac{1}{2}\phi_i g_{ij}^{-1}\phi_j-i\phi_i\rho_i+\psi_i^{\dagger}(-i\sigma_3\partial_x+\sigma_0\partial_\tau)_{ij}\psi_j\right] \tag{3.2}$$

The conserved quantity is thus hidden deeply in the implicit formula above. Let us recall that the theory is from relativistic Hamiltonian and have some intrinsic symmetries. Introducing $x_1=\tau$, $x_2=x$, $dx=dx_1dx_2$ and the 'Dirac Spinor' $\psi=(\psi_+,\psi_-)^T,\bar{\psi}=(\sigma_1\psi)^\dagger=\left(\psi_-^\dagger,\psi_+^\dagger\right)$ together with the re-definition of the auxiliary fields:$\phi_1=\frac{1}{2}\left(\phi_++\phi_-\right),\phi_2=\frac{1}{2i}\left(\phi_1-\phi_2\right)$,we find:

$$S\left[\bar{\psi},\psi,\phi\right]=\frac{1}{2}\int d^2x\phi^T g^{-1}\phi-\int d^2x\bar{\psi}\left(\sigma_\mu\partial_\mu-i\sigma_\mu\phi_\mu\right)\psi \tag{3.3}$$

Where the auxiliary fields come into the action with a form of two-dimensional vector potential.

Back to our model, integrate out the matter field, we arrive at the effective action:

$$S_{eff}\left[\phi\right]=\frac{1}{2}\int d^2x\phi^T g^{-1}\phi-\mathrm{Tr}\ln\mathcal{G}^{-1}\left[\phi\right] \tag{3.4}$$

Where $\mathcal{G}^{-1}\left[\phi\right]=\sigma_\mu\partial_\mu-i\sigma_\mu\phi_\mu$

By functional expansing the Trln term, we obtain:

$$Tr\ln\mathcal{G}^{-1}\left[\phi\right]=\underbrace{Tr\ln\left(\sigma_\mu\partial_\mu\right)}_{\to 0}-i\underbrace{Tr\left(\left(\sigma_\mu\partial_\mu\right)^{-1}\sigma_\mu\phi_\mu\right)}_{0}+\frac{1}{2}\underbrace{Tr\left(\left(\sigma_\mu\partial_\mu\right)^{-1}\sigma_\mu\phi_\mu\left(\sigma_\mu\partial_\mu\right)^{-1}\sigma_\mu\phi_\mu\right)}_{\neq 0} \tag{3.5}$$

The first term is zero due to the zero Tr result while the second term zero due to summation of an odd function. We only expand the Green function up to second order. This is ensured by so-called **closed loop theorem**. And this equality is actually exact in 1D Luttinger model. One can basically picture this by considering the bosonized Hamiltonian is known to be quadratic and thus the expansion truncates at the second order. Some people are credited to prove this theorem, but I cannot find the original paper in website anymore[3].

Take the Fourier transformation and calculate the matrix in Tr term carefully:

$$\begin{aligned}&Tr\left(\left(\sigma_\mu\partial_\mu\right)^{-1}\sigma_\mu\phi_\mu\left(\sigma_\mu\partial_\mu\right)^{-1}\sigma_\mu\phi_\mu\right)\\&=\sum_{p,k}\left(\sigma_\mu p_\mu\right)^{-1}\sigma_\mu\phi_{\mu,-k}\left(\sigma_\mu(p+k)_\mu\right)^{-1}\sigma_\mu\phi_{\mu,k}\end{aligned} \tag{3.6}$$

Treat the summation as the integral and $p=(\epsilon,p),k=(w,k)$, we arrive at:

$$\begin{aligned}&S_{eff}\left[\phi\right]=\frac{1}{2}\int d^2x\phi^T g^{-1}\phi-\frac{1}{2}Tr\left(\left(\sigma_\mu\partial_\mu\right)^{-1}\sigma_\mu\phi_\mu\left(\sigma_\mu\partial_\mu\right)^{-1}\sigma_\mu\phi_\mu\right)\\&=\frac{T}{2L}\sum_k\phi_{-k,i}\left(g_{ij}^{-1}+\delta_{ij}\Pi_{i,-k}\right)\phi_{k,j}\\&\Pi_{i,-k}=\int\frac{dpd\varepsilon}{4\pi^2}\frac{1}{\left(\varepsilon+isp\right)\left(\varepsilon+\omega+is\left(p-k\right)\right)}\end{aligned} \tag{3.7}$$

[3]Most people cite it as: T. Bohr, Nordita preprint 81/4, Lectures on the Luttinger Model, 1981 (unpublished).

Where $i, j = \{+, -\}$ again. In the kernel matrix, the integral is divergent. To regularize this, we introduce a cutoff Λ in momentum and then:

$$\Pi_{i,-k} = \int_{-\Lambda}^{\Lambda} \frac{dpd\varepsilon}{4\pi^2} \frac{1}{(\varepsilon + isp)(\varepsilon + \omega + is(p-k))} = \frac{1}{2\pi} \frac{k}{is\omega + k} \tag{3.8}$$

Now, the system is exactly solved and we want to know if it can hold a bosonic representation.

4 Dictionary of Bosonization

Now we can go back to check the essential points of the whole derivation. Note that the fermionic theory have its natural particle-hole excitations and bosonization procedure just depends on those density fluctuations. However, the transformation from fermionic operators to bosonic ones is universal; even if the original system does not have well-defined particle-hole excitations. Basically, we can take the transformation and have a trial if the correlation function is more computable in the new representation. So, what will we get from the particle hole symmetry? It's not very clear to me now, but I suspect that the particle hole excitations ensure the renormalization properties of a theory as Eq(1.4) have shown to us.

4.1 Review of scalar field theory

We first take a review of 1D scalar field theory and then we can point out explicitly those relation.

The scalar field have the Hamiltonian density as:

$$\mathcal{H} = \frac{\pi^2}{2} + (\partial_x \theta)^2 \tag{4.1}$$

Where the two operator is canonical and obey the relation $[\theta(x), \pi(y)] = i\delta(x-y)$ and thus the theory can be mapped to operators explicitly:

$$\begin{aligned} \theta(x) &= \int_{-\infty}^{+\infty} \frac{dpe^{-\frac{1}{2}\alpha|p|}}{\sqrt{2|p|}2\pi} \left[b_p e^{ipx} + h.c.\right] \\ \Pi(x) &= \int_{-\infty}^{+\infty} \frac{dpe^{-\frac{1}{2}\alpha|p|}}{\sqrt{2|p|}2\pi} \left[-ib_p e^{ipx} + h.c.\right] \end{aligned} \tag{4.2}$$

Where the parameter α vanishes faster than any parameters in the model and the mode operators $b_p, b_p^\dagger$ obey $[b_p, b_{p'}^\dagger] = 2\pi\delta(p-p')$, which leads to the relation:

$$[\theta(x), \Pi(y)] = \frac{i\alpha/\pi}{\alpha^2 + (x-y)^2} \overset{\alpha \to 0}{\sim} i\delta(x-y) \tag{4.3}$$

The Hamiltonian density becomes:$\mathcal{H} = |p| b_p^\dagger b_p$.

Then, we define the left or right operator as:

$$\theta_\pm(x) = \frac{1}{2}\left[\theta(x) \pm \int_{-\infty}^{x} dx' \Pi(x')\right] = \pm \int_0^{\pm\infty} \frac{dpe^{-\frac{1}{2}\alpha|p|}}{\sqrt{2|p|}2\pi} \left[b_p e^{ipx} + h.c.\right] \tag{4.4}$$

With the relation $[\theta_\pm(x), \theta_\pm(y)] = \pm\frac{i}{4}\,\mathrm{sgn}(x-y), [\theta_+(x), \theta_-(y)] = \frac{i}{4}$
Thus, we have the relation

$$\begin{aligned}
\Phi(x) &= \int_{-\infty}^{x} dx'\Pi(x') = \theta_+ - \theta_- \\
\theta(x) &= \theta_+ + \theta_-
\end{aligned} \tag{4.5}$$

4.2 Bosonic-Fermionic Duality

We now try to represent fermionic operators in terms of bosonic ones, i.e.,

$$\psi_\pm(x) = \frac{1}{\sqrt{4\pi\alpha}} e^{i\sqrt{4\pi}\theta_\pm(x)} \tag{4.6}$$

When expanding this using the definition Eq(4.4), one can find that the field Π serves as a Jordan string.

Then, one can derive several fermionic quantities to their bosonic dualities, for example and most importantly, those covariant bilinears,

$$\boxed{\begin{aligned}
&\bar{\psi}\psi = \psi^\dagger\sigma_2\psi = -i\psi_+^\dagger\psi_- + h.c. = -\frac{1}{\pi\alpha}\cos\sqrt{4\pi}\theta \\
&\bar{\psi}i\gamma^5\psi = \frac{1}{\pi\alpha}\sin\sqrt{4\pi}\theta \\
&j^\mu = \bar{\psi}\gamma^\mu\psi = \frac{\varepsilon^{\mu\nu}}{\sqrt{\pi}}\partial_\nu\theta \\
&\bar{\psi}\partial\psi = \frac{1}{2}(\partial^\mu\theta)^2
\end{aligned}} \tag{4.7}$$

Note that, the locality of current operators remain intact but lose its quadric form. However, they still have some common features that are some combination of $\exp(U)$First two equations.. Recalling that we have derived a beautiful conclusion that connects the correlation function of $< \exp >$ to $\exp(<>)$, i.e.,

$$\langle\exp(iU)\rangle = \exp\left(\left\langle -\frac{1}{2}U^2\right\rangle\right) \tag{4.8}$$

We will discuss this in next section in details.

One may be confused that the last two equation when the derivatives of bosonic fields arise, i.e, the *point splitting* tricks:

$$\begin{aligned}
&\psi_+^\dagger(0)\psi_+(0) = \lim_{x\to 0}\frac{1}{2\pi\alpha}e^{-i\sqrt{4\pi}\theta_+(x)}e^{i\sqrt{4\pi}\theta_+(0)} \\
&= \lim_{x\to 0}\frac{1}{2\pi\alpha} : e^{-i\sqrt{4\pi}\theta_+(x)}e^{i\sqrt{4\pi}\theta_+(0)} : e^{4\pi\mathcal{G}_+(x)} \\
&= \lim_{x\to 0} : 1 - i\sqrt{4\pi}x\partial_x\theta_+ : \frac{i}{2\pi(x+i\alpha)} \\
&= \lim_{x\to 0}\frac{i}{2\pi x} + \frac{1}{\sqrt{\pi}}\partial_x\theta_+
\end{aligned} \tag{4.9}$$

4.3 Essential correlation function(CF)

In a quantum theory, the observables are always encoded in the correlation functions. The fermionic version is evaluated in the ground state locally where some of them are highly nonlocal in bosonic theory. In this section, we develop a solution of this essential CFs, and thus we can compute all the observable in corresponding fermionic theory by the tricks introduce above.

As an introductory, we first calculate some bosonic CFs learned from QFT. First, we compute $\mathcal{G}_{\pm}(x) = \left\langle \theta_{\pm}(x)\,\theta_{\pm}(0) - \theta_{\pm}^2(0) \right\rangle$ and,

$$\begin{aligned} \mathcal{G}_{+}(x) &= \int_0^\infty \frac{dp e^{-\frac{1}{2}|\alpha| p}}{2\pi\sqrt{2p}} \int_0^\infty \frac{dq e^{-\frac{1}{2}|\alpha| q}}{2\pi\sqrt{2q}} \left\langle b(p)\, b^\dagger(q) \right\rangle \left(e^{ipx} - 1\right) \\ &= \frac{1}{4\pi} \int_0^\infty dq \frac{\exp(iqx) - 1}{q \exp(\alpha q)} \\ &= \frac{1}{4\pi} \ln \frac{\alpha}{\alpha - ix} \end{aligned} \tag{4.10}$$

The associated integral is evaluated in appendix. We thus have:

$$\mathcal{G}_{\pm}(x) = \frac{1}{4\pi} \ln \frac{\alpha}{\alpha \mp ix} \tag{4.11}$$

And we can prove,

$$\begin{aligned} \mathcal{G}(x) &= \left\langle \theta(x)\,\theta(0) - \theta^2(0) \right\rangle \\ &= \mathcal{G}_{+}(x) + \mathcal{G}_{-}(x) + 2\left\langle \left[\theta_{+}(x) - \theta_{+}(0)\right] \theta_{-}(0) \right\rangle \\ &= \mathcal{G}_{+}(x) + \mathcal{G}_{-}(x) \\ &= \frac{1}{4\pi} \ln \frac{\alpha^2}{\alpha^2 + x^2} \end{aligned} \tag{4.12}$$

Those quantity can directly lead to the result:

$$\begin{aligned} \mathcal{G}_{\beta}^{\pm}(x) &= \left\langle e^{i\beta\theta_{\pm}(x)} e^{-i\beta\theta_{\pm}(0)} \right\rangle \\ &= \left\langle : e^{i\beta\theta_{\pm}(x)} e^{-i\beta\theta_{\pm}(0)} : \right\rangle e^{\beta^2 \left\langle \theta_{\pm}(x)\theta_{\pm}(0) - \frac{\theta_{\pm}{}^2(x) + \theta_{\pm}{}^2(0)}{2} \right\rangle} \\ &= e^{\beta^2 \frac{1}{4\pi} \ln \frac{\alpha}{\alpha \mp ix}} = \left(\frac{\alpha}{\alpha \mp ix} \right)^{\frac{\beta^2}{4\pi}} \end{aligned} \tag{4.13}$$

The same spirit will cause the evaluation of the correlation function:

$$\begin{aligned} \mathcal{G}_{\beta}(x) &= \left\langle e^{i\beta\theta(x)} e^{-i\beta\theta(0)} \right\rangle \\ &= \left\langle : e^{i\beta\theta(x)} e^{-i\beta\theta(0)} : \right\rangle e^{\beta^2 \left\langle \theta(x)\theta(0) - \frac{\theta^2(x) + \theta^2(0)}{2} \right\rangle} \\ &= e^{\beta^2 \frac{1}{4\pi} \ln \frac{\alpha^2}{\alpha^2 + x^2}} = \left(\frac{\alpha}{\alpha^2 + x^2} \right)^{\frac{\beta^2}{4\pi}} \end{aligned} \tag{4.14}$$

Which we have met in the Luttinger model.

An important identity which is a good exercise and we will use in the next section is :

$$
\begin{aligned}
&\left[\frac{-1}{\pi\alpha}\cos\sqrt{4\pi}\theta\left(x_0\right)\right]^2 \\
&= \frac{1}{4\pi^2\alpha^2}\lim_{x\to x_0}\left[e^{i\sqrt{4\pi}\theta(x)}+cc\right]\left[e^{i\sqrt{4\pi}\theta(x_0)}+cc\right] \\
&= \frac{1}{2\pi^2\alpha^2}\lim_{x\to x_0}\left[\cos\sqrt{4\pi}\left(\theta\left(x\right)+\theta\left(x_0\right)\right)+\cos\sqrt{4\pi}\left(\theta\left(x\right)-\theta\left(x_0\right)\right)\right] \\
&= \frac{1}{2\pi\alpha^2}\cos\sqrt{16\pi}\theta\left(x_0\right)+\frac{1}{2\pi^2\alpha^2}\lim_{x\to x_0}\left[\cos\sqrt{4\pi}\left(\theta\left(x\right)-\theta\left(x_0\right)\right)\right]
\end{aligned}
\tag{4.15}
$$

To evaluate the second term we use the tricks Eq A.10, and we arrive at:

$$
\begin{aligned}
&\frac{1}{2\pi^2\alpha^2}\lim_{x\to x_0}\left[\cos\sqrt{4\pi}\left(\theta\left(x\right)-\theta\left(x_0\right)\right)\right] \\
&= \lim_{x\to x_0}\frac{1}{2\pi^2\alpha^2}\left[:\cos\sqrt{4\pi}\left(\theta\left(x\right)-\theta\left(x_0\right)\right):\frac{\alpha^2}{\left(x-x_0\right)^2+\alpha^2}\right] \\
&= \lim_{x\to x_0}\frac{1}{2\pi^2\alpha^2}\left(1-\frac{\left(x-x_0\right)^2}{2}\left(4\pi\right)\left(\partial_x\theta\left(x_0\right)\right)^2\right)\frac{\alpha^2}{\left(x-x_0\right)^2+\alpha^2} \\
&= -\frac{1}{\pi}\left(\partial_{x_0}\theta\right)^2+const
\end{aligned}
\tag{4.16}
$$

Thus, we have:

$$
\begin{aligned}
&\left[\frac{-1}{\pi\alpha}\cos\sqrt{4\pi}\theta\left(x\right)\right]^2 \\
&= \frac{1}{2\pi\alpha^2}\cos\sqrt{16\pi}\theta\left(x\right)-\frac{1}{\pi}\left(\partial_x\theta\right)^2
\end{aligned}
\tag{4.17}
$$

Which leads to:

$$
\boxed{\left(\bar{\psi}\psi\right)^2=\frac{1}{2\pi\alpha^2}\cos\sqrt{16\pi}\theta\left(x\right)-\frac{1}{\pi}\left(\partial_x\theta\right)^2}
\tag{4.18}
$$

5 Application: FQH with SC — Sine Gordan Model and its RG

In this section a novel application of bosonization technique is shown. Now we know that this method is very powerful to solve 1D problem. Recent years have witnessed the progress of novel edge state of fractional quantum hall effect. This effect is based on the interacting 2D electronic system and developed Chern-Simons field theory. It was proved this system holds chiral luttinger-model on its edge and after coupling with a superconductor this system becomes an ideal platform for topological quantum computation. We first review some essential knowledge about fractional quantum hall effect and then use the learned method to develop an effective theory to describe the phenomenon between excitations on different edges of a QHE

5.1 Step1: Construction of Chern-Simons Theory

Most of the books choose to construct Chern-Simons theory phenomenally. Here, we want to present a rigorous derivation *ab initio*. Though it's a bit tedious, we can get some exercises of the path integral formula. The problem is 2D interacting electrons subject to a perpendicular magnetic field of strength in an infinite homogenous plane(we then consider a finite geometry). The Hamiltonian is $H = H_0 + H_{int}$, where

$$\begin{aligned} H_0 &= \int d^2 x a^\dagger(\vec{x}) \left[\frac{1}{2m} (-i\partial + A_{ext})^2 + V(\vec{x}) \right] a(\vec{x}) \\ H_{\text{int}} &= \frac{1}{2} \int d^2 x d^2 x' \left(\rho(\vec{x}) - \rho_0 \right) V(\vec{x} - \vec{x}') \left(\rho(\vec{x}') - \rho_0 \right) \end{aligned} \tag{5.1}$$

and, $A_{ext} = \frac{B_{ext}}{2}(y, -x)^T$ is the vector potential in the symmetric gauge.

In many introductory book, one can find many debates on the concept of anyons, which can be achieved in this platform by the idea of composite fermions. In this idea, $2s\phi$ phases are attached to a bare electron, creating a perpendicular magnetic field at the coordinates of the electrons, i.e, $b = \varepsilon^{ij}\partial_i a_j = -4\pi s \sum\limits_i \delta(\vec{x} - \vec{x}_i)$ where $\partial_i = \partial_{\vec{x}_i}$ thus leading to the vector potential $\vec{a} = -2s\vec{\partial} \sum\limits_i \arg(\vec{x} - \vec{x}_i)$ and the Pierls substitution:

$$\Psi \to \Psi(x_1, ..., x_N) \exp\left[-2is \sum_{i<j} \arg(\vec{x}_i - \vec{x}_j) \right] \tag{5.2}$$

Within the framework of the second quantization, the transformation is:

$$a \to a \exp\left[2is \int d^2 x' \arg(\vec{x} - \vec{x}') \rho(\vec{x}') \right] \tag{5.3}$$

Inserting this into the original Hamiltonian, one obtains:

$$H \to \int d^2 x a^\dagger(\vec{x}) [\frac{1}{2m}(-i\partial_{\vec{x}} + A)^2 + V(x)] a(\vec{x}) \tag{5.4}$$

Where the total gauge field $A = A_{ext} + a$ and

$$a(\vec{x}) = -2s \int d^2 x' \frac{(x_1 - x_1')\vec{e_2} - (x_2 - x_2')\vec{e_1}}{|\vec{x} - \vec{x}'|^2} \rho(x') \tag{5.5}$$

Thus, we immediately see the guage theory nature of this Hamiltonian[4]. We then want to quantize the theory by normal procedure (FP functional integral, and the rogorous derivation is remained for readers). First, fixing the gauge as $\partial_i a_i = 0$, we denote the configuration that manifest this as $a_\perp$ thus we have $1 = \mathcal{N} \int \mathcal{D}\vec{a}_\perp \prod_{\vec{x},t} \delta(b(\vec{x}, t) + 4\pi s \rho(\vec{x}, t))$ where $b = \epsilon^{ij}\partial_i a_{\perp,j}$. We have the action $S = S_0 + S_{int}$:

$$\begin{aligned} S_0[\bar{\psi}, \psi] &= \int dt d^2 x \bar{\psi} [i\partial_t + \mu - \frac{1}{2m}(-i\partial_{\vec{x}} + \vec{A}[\bar{\psi}, \psi])^2 - V(x)] \psi \\ S_{int}[\bar{\psi}, \psi] &= -\frac{1}{2} \int dt d^2 x d^2 x' (\rho(\vec{x}) - \rho_0) V(\vec{x} - \vec{x}') (\rho(\vec{x}') - \rho_0) \end{aligned} \tag{5.6}$$

[4]For the reason of constructing the Hamiltonian like this, see, for example, A. M. Polyakov, Fermi-Bose Transmutations Induced by Gauge Fields, Modern Physics Letters A, Vol 3, No.3(1988)

And the partition function is :

$$
\begin{aligned}
\mathcal{Z} &= \mathcal{N}\int \mathcal{D}(\bar{\psi},\psi)\mathcal{D}\vec{a}_\perp \prod_{\vec{x},t}\delta(b(\vec{x},t)+4\pi s\rho(\vec{x},t))\exp(iS) \\
&= \mathcal{N}\int \mathcal{D}(\bar{\psi},\psi)\mathcal{D}\vec{a}_\perp \mathcal{D}\phi \exp(iS - i\int d^2x dt \phi(\frac{b}{4\pi s}+\rho)) \\
&= \mathcal{N}\int \mathcal{D}(\bar{\psi},\psi)\mathcal{D}\vec{a}_\perp \mathcal{D}\phi \exp(iS_{CF}[\psi,\bar{\psi},\vec{a}_\perp,\phi] + i\frac{\theta}{2}S_{CS}[\vec{a}_\perp,\phi])
\end{aligned}
\tag{5.7}
$$

Where we use the fact that the functional Fourier transformation of constant is the functional delta function and $\theta = \frac{1}{2\pi s}$, and the total action is divided into two parts:

$$
\begin{aligned}
S_{CF} &= \int d^2x dt \bar{\psi}(i\partial_t + \mu - \phi + \frac{1}{2m}(-i\partial_{\vec{x}} + \vec{A})^2 - V)\psi + S_{int} \\
S_{CS}[a_\perp] &= -\int d^2x dt \phi \epsilon_{ij}\partial_i a_{\perp,j}
\end{aligned}
\tag{5.8}
$$

From the observation of the structure of the action, the theory imply one of fermions coupled to a $2+1$-D gauge potential $a_\perp = (\phi, \vec{a}_\perp)$ but the contribution S_{CS} is not gauge invariant for we have fixed the theory to a specific gauge $\partial_i a_i = 0$. Extending this to a gauge-invariant version, one can obtain the Chern-Simons action:

$$
\boxed{S_{CS}[a] = -\int d^3x \epsilon_{\mu\nu\sigma} a_\mu \partial_\nu a_\sigma}
\tag{5.9}
$$

Where $x_\mu = (x_0, x_1, x_2)$ and $\partial_\mu = (-\partial_0, \partial_1, \partial_2)$ and contributing to the partition function with a coefficient $\frac{\theta}{4}$. And the composited fermion term reads as:

$$
\boxed{S_{CF} = \int d^2x dt \bar{\psi}(i\partial_t + \mu - \phi + \frac{1}{2m}(-i\partial_{\vec{x}} + \vec{A}_{ext} - \vec{a})^2 - V)\psi + S_{int}}
\tag{5.10}
$$

When fixing the gauge as $a_\perp = (\phi, \partial_2 f, -\partial_1 f)$, one can recover the 2-D electron gas in a gauge potential.

If this extension method cannot fully convince you the validity of the theory, one can simply begin with a non-interacting classical system, denoted by $\mathcal{L}$ with the conserved current j^μ and then add the Chern-Simons term and the $j^\mu a_\mu$ term where a_μ is defined in previous content. It comes with a exercise that the corresponding Hamiltonian is itself $H = \frac{1}{2}\sum_i (p_i - qa(x_i))^2$ which is exactly what we need. One may be confused that, the Hamiltonian seems lacking the gauge invariance from the observation of the absence of qa_0 term. The answer is that the expression of a_μ is highly nonlocal thus when deriving the equation of motion, some additional terms appear. There are many discussions of the physical content of this action.[5]

[5] e.g, David Tong's Lecture note: http://www.damtp.cam.ac.uk/user/tong/qhe.html Wilzcek, Fractional Statistics and Anyon superconductivity, World Scientific.

5.2 Step 2: Effective Field Thoery: Mean Field and Fluctuation

However, the action including both CF and CS term is very complicated. One would like to obtain an effective field theory to describe the low energy excitations. [6]

Invoking the Hubbard-Stratonovich transformation for decoupling the interaction part with the auxiliary field $\sigma(x)$ and $x = (x_0, \vec{x})$:

$$\exp(iS_{int}) = \int \mathcal{D}\sigma \exp \begin{bmatrix} \frac{i}{2}\int d^3xd^3x'\sigma(x)[\delta(x_0 - x_0')V^{-1}(\vec{x} - \vec{x}')]\sigma(x') + \\ i\int d^3x(\rho(x) - \rho_0)\sigma(x) \end{bmatrix} \tag{5.11}$$

with a normalization constant implied in the definition of the measure of $\mathcal{D}\sigma$.

Integrate out the fermionic degree of freedom in CF term, we obtain:

$$\begin{aligned} S[a,\sigma] = &-i \ \mathrm{trln}\mathcal{G}^{-1} + \frac{\theta}{4}S_{CS}[a] - \rho_0\int d^3x\sigma(x) \\ &+ \frac{1}{2}\int d^3xd^3x'\sigma(x)V^{-1}(\vec{x} - \vec{x}')\delta(x_0 - x_0')\sigma(x') \end{aligned} \tag{5.12}$$

Where

$$\mathcal{G}^{-1} = i\partial_0 + \mu - \phi - \sigma + \frac{1}{2}(-i\nabla + \vec{A})^2 - V \tag{5.13}$$

We thus seek the solution of the classical equation of motions:

$$\frac{\delta S[a,\sigma]}{\delta a_\mu(x)} = \frac{\delta S[a,\sigma]}{\delta \sigma(x)} = 0 \tag{5.14}$$

and we denote the solution as $\bar{\sigma}$ and $\bar{a}$ with $\bar{b} = \epsilon_{ij}\partial_i\bar{a}_j$.For the variation of the trln term, a useful formula is $(\delta/\delta f)\mathrm{trln}G^{-1} = \mathrm{tr}(G\delta/\delta f G^{-1})$ while because the theory is quadric, variation with respect to any field is Euler-lagragian type.

One can simply variate with respect to $a_0 = \phi$ and obtain (or, as your experience in deriving Maxwell equation):

$$\bar{\rho} = i\mathcal{G}(\bar{a},\bar{\sigma}) = \frac{\theta}{2}\bar{b} \tag{5.15}$$

And variation with respect to the vector component of a gives nothing but other electromagnet equation and the continuity equation, e.g, $\vec{J}^i = \frac{\theta}{2}\epsilon_{ij}E_j$, where $\vec{J} = i\mathrm{tr}[G(-i\nabla + \vec{A})]$. Together, we have $j_\mu = \frac{\theta}{4}\epsilon_{\mu\nu\lambda}F_{\nu\lambda}$

The section equation (sketched out in appendix) then gives:

$$\sigma(x) = -\int d^3\delta(x_0 - x_0')V(\vec{x} - \vec{x}')(\rho(x') - \rho_0) \tag{5.16}$$

.

The physical content of this mean field is also intuitive: the potential created by local density fluctuations compensates the interaction potential in effective theory.

[6]See, for example, Ana Lopez and E. Fradkin, Fractional Quantum Hall Effect and Chern-Simons Gauge Theories.

These equation may lead to many type of solution, e.g, liquid state adn Wigner crystals. We present a different one here. When the external potential $V(\vec{x})$ vanishes, the auxiliary field $\sigma(x)$ vanishes, too. Together with the mean-field equation $\rho[\bar{a}, 0] = \rho_0 =$ const. and $\phi = 0$, i.e, no electric field exists. Thus,

$$\bar{B} = \frac{2}{\theta}\rho_0 = 4\pi s\rho_0 \tag{5.17}$$

These lead to a fact that :

$$\vec{a} = (2s)\vec{A}_{ext}\{ \text{ a function of chemical potential and N } \} \tag{5.18}$$

The function of chemical potential and N, denoted by ν, have the meaning of the filling fraction in assumed Landau lever picture and not generally integer(if the reduced picture is gapless) . However, we are confident about the gap because of the success of the Laughlin's argument(or i.e, the experiment suggests a gapped excitation.). Then we have $\vec{a} = (2s\nu)\vec{A}_{ext}$ where ν is a filling fractions. That means directly that there are p composited (reduced) Landau levels are fully occupied will the filling fractions ν. The effective field controlling the Landau level is $B = B_{ext} - b$. By the fact that $b = 4\pi sNL^{-2}$ and $\Phi_{eff} = BL^2 = 2\pi N/p$, we have:

$$\frac{2\pi N}{p} = BL^2 = (B_{ext} - 4\pi sNL^{-2})L^2 \tag{5.19}$$

which leads to:

$$\boxed{\nu = \frac{2\pi N}{B_{ext}L^2} = (\frac{1}{p} + 2s)^{-1} = \frac{p}{2sp+1}} \tag{5.20}$$

The mean field theory with some self-consistency checks can give the expectation results:

The FQHE can be interpreted as an IQHE, but of composited fermions.

We are now proceeding to construct construct the effective theory based on this mean field theory. Expanding $a = \bar{a} + \delta a$ and $\sigma = \bar{\sigma} + \delta\sigma$, we need to further expand the action to second order in the deviations of $(\delta a, \delta\sigma)$. Setting $a_0 = \phi + \delta\sigma$ will consume the final degree of freedom of the system and highly save our hair. The picked shifting removes σ from the green function and contributed an additional term in CS action: $S_{CS}[\bar{a} + \delta a] = S_{CS}[\bar{a} + \delta a] - 2\int d^3x \delta\sigma(x)\delta b(x)$. This leads to the total $\delta\sigma$ expansion:

$$S[a, \sigma] \simeq S[a] - \frac{\theta}{2}\int d^3x \delta\sigma\delta b + \frac{1}{2}\int d^3x d^3x' \delta\sigma V^{-1}(\vec{x} - \vec{x}')\delta(x_0 - x_0')\delta\sigma(x') \tag{5.21}$$

Integrating over $\delta\sigma$ we obtain:

$$S[a] + \frac{1}{2}\int d^3x d^3x' b(x) V(\vec{x} - \vec{x}')\delta(x_0 - x_0')b(x') \tag{5.22}$$

The physical interpretation of the induced term is the fluctuations of the density will be tied to the statistical guage field. We first focus on the first term for the electromagnetism response is of interest.

By adding an external term A to the action, and expanding the action as $S[a, A] = \Sigma_n S^{(n)}[a, A]$ where $S^{(n)}[a, A]$ is of total order n in δa and A. The zeroth-order term is mean-field term and not of interest. The first-order term do not contain δa but an in essential term $iA_\mu \bar{j}^\mu$ because the mean-field density $\bar{j}^0$ is trivial and the vector $\vec{\bar{j}}$ vanishes(see Eq (5.17)). At the second-order we have:

$$S^{(2)}[a,A] = \frac{1}{2}\int d^3x d^3x' (\delta a + A)_\mu(x) K_{\mu\nu}(x,x')(\delta a + A)_\nu(x') + \frac{\theta}{4} S_{CS}[a] \quad (5.23)$$

where $K_{\mu\nu}(x,x')$ is the the linear response kernel defined as $K_{\mu\nu}(x,x') = \frac{\delta^2 S[CF]}{\delta a_\mu \delta a_\nu}$ and evaluate in mean-field level $a = \bar{a}$.

Due to the general properties of linear response kernel, we can however construct our theory in an explicit way due to some gauge invariance consideration:

For the Fourier transformed linear response $K(q)$, this function can be expanded in powers of q. Thus the action can be expanded in forms of $S^{(2)}[a,A] = \Sigma_l S^{(2,l)}[a,A]$ where $S^{(2,l)}[a,A]$ contains l th order in derivatives ∂_i in the kernel. As the system is gapped in an insulating phase, if we were in $3-D$ case, no $S^{(2,1)}$ term can be constructed. However, in $2-D$ case, there exists other gauge-invariant first-order derivative term, that is, CS term again. We thus conclude that :

$$S^{2,1}[a,A] = cS_{CS}[\delta a + A] + \frac{\theta}{4} S_{CS}[a] \quad (5.24)$$

With the Fourier transformed CS term: $S_{CS}[t] = -\int d^3x \epsilon_{\mu\nu\sigma} t_\mu \partial_\nu t_\sigma = -i\Sigma_q \epsilon_{\mu\nu\sigma} t_\mu(-q)\partial_\nu t_\sigma(q)$ Compare this with Eq5.23, we have

$$K_{\mu\nu}(q) = i2c\epsilon_{\mu\nu\sigma} q_\sigma \quad (5.25)$$

With the experience of quantum hall effect of composited fermions, we have $\sigma_{xy}^{(CF)} = -i \lim_{q\to 0} K_{xy}\omega, \vec{q} = \frac{p}{2\pi}$ in the mean-field level. These lead to $c = \frac{sigma_{xy}}{2}$ and thus:

$$S^{2,1}[a,A] = \frac{p}{4\pi} S_{CS}[\delta a + A] + \frac{\theta}{4} S_{CS}[\delta a] \quad (5.26)$$

Integrating the fluctuation field out we have:

$$S_{eff} = \frac{\sigma_{xy}}{2} S_{CS}[A] \quad (5.27)$$

and $\sigma_{xy} = (\frac{1}{\sigma_{xy}^{(CF)}} + \frac{2}{\theta})^{-1}$ and

$$\sigma_{xy} = \frac{1}{2\pi}\frac{p}{2sp+1} \quad (5.28)$$

Which is consistent with the qualitative description developed last subsection.

Together with the invariant electromagnetic term, the final effective theory is:

$$S_{eff} = \frac{\sigma_{xy}}{2} S_{CS}[A] + \frac{1}{2}\int d^3x (\epsilon \vec{E}\cdot\vec{E} + \chi b^2) + \cdots \text{(invariant high-order term)} \quad (5.29)$$

5.3 Step 3: Excitations on the edge: Chiral Luttinger Model

We have constructed the effective theory in the bulk. Now we can develop the excitation theory on the edge. Before formal derivation of this theory, we first construct this with our familiar phenomenal method.

Consider a electron liquid (that is, a system with a bulk that support no gapless excitation and i.e incompressible), we can parameterize the boundary with a height function$h(x)$ and the density profile with $\rho(x)$. We have $\rho(x)dx = \frac{\nu B}{2\pi}\int_0^{h(x)} dx_\tau dx = \frac{\nu B}{2\pi}h(x)dx$. Then, because of the confining potential on the edge:$V = Ex_\perp$, a distortion on the boundary costs a energy :

$$H = \int dx \int_0^{h(x)} dx_\tau \sigma E x_\tau = \frac{\nu EB}{4\pi}\int dx h^2(x) = \frac{E}{4\pi\nu B}\int dx(2\pi\rho)^2 \tag{5.30}$$

Define the transverse field as $\partial_x\phi = 2\pi\rho$, we have the Hamiltonian of interest: $H = \int dx \frac{E}{4\pi\nu B}(\partial_x\phi)^2$ with the equation of motion compatible with the continuity equation $\partial_t\phi = v\partial_x\phi$ where $v = \frac{E}{2\pi B}$. Thus the we have the Fourier transformed Hamiltonian $H = \frac{v}{2\pi\nu}\Sigma_k k^2\phi_k\phi_{-k}$ and the canonical conjugation $\pi_k = \frac{1}{2\pi\nu}(-ik)\phi_{-k}$ and related quantum commutation relation. Return real space and we can get the action of the system: $S = \int d\tau(H - i\pi\partial_\tau\phi) = \frac{1}{2\pi\nu}\int d\tau dx(v(\partial_x\phi)^2 - i\partial_x\phi\partial_\tau\phi)$ which is the action of a chiral Luttinger liquid for the corresponding equation of motion have the solution of form $\rho \sim g(x+vt)$ but $g(x-vt)$ is not a solution.

Next, we want to develop a rigorous theory from ab-initio consideration. The only theory at hand is the effective Chern-Simons theory which is gauge invariant on the bulk. However, one will challenge the gauge invariance of the finite size sheet(e.g, a square sheet with an edge parameterized as x, see appendix for details). As we have done in phenomenal method, we first make a variation of the fields which gives:

$$\begin{aligned}\delta S_{CS} &= -\frac{\sigma}{2\pi}\int d^3x\epsilon_{\mu\nu\sigma}[\delta a_\mu\partial_\nu a_\sigma + a_\mu\partial_\nu\delta a_\sigma] \\ &= \frac{\sigma}{2\pi}\int d^3x\epsilon_{\mu\nu\sigma}[\delta a_\mu F_{\mu\sigma} + \partial_\mu(a_\nu\delta a_\sigma)]\end{aligned} \tag{5.31}$$

To obtain the right equation of motion $F_{\mu\sigma} = 0$ only if we can set the last term to zero. To this end, we can set $a_0(x, y = 0) = 0$ or $a_x(x, y = 0) = 0$ or we can take a linear combination of these :

$$(a_t - va_x) = 0 \qquad \text{when} \quad y = 0 \tag{5.32}$$

To verify the gauge invariance of the Chern-simons field, we have:

$$\begin{aligned}
S[a_\mu + \partial_\mu f] - S[a_\mu] &= -\int d^3x \epsilon_{\mu\nu\sigma}(a_\mu + \partial_\mu f)\partial_\nu(a_\sigma + \partial_\sigma f) - a_\mu \partial_\nu a_\sigma \\
&= -\int d^3x \epsilon_{\mu\nu\sigma}(a_\mu \partial_\nu \partial_\sigma f + \partial_\mu f \partial_\nu a_\sigma + \partial_\mu f \partial_\mu \partial_\sigma f) \\
&= -\int d^3x \epsilon_{\mu\nu\sigma}\partial_\mu f \partial_\nu a_\sigma \qquad (5.33) \\
&= -\int d^3x \epsilon_{\mu\nu\sigma}\partial(f\partial_\nu a_\sigma) \\
&= -\int d\sigma dx_0 f(F_{0\sigma})
\end{aligned}$$

Where in the second equation we have used $\partial_\mu \partial_\nu = \partial_\nu \partial_\mu$ and in the last line we have used the stokes' theorem and where σ parameterize the edge so that $\sigma \leftrightarrow x$ if we want to have a theory in any other geometry.

This means the Chern-Simons theory lost its gauge invariance which can be fixed by defining the gauge transformation as $f(y=0)=0$ but this will erase the gauge degree of freedom. So, we try to extend the boundary condition into the bulk, by requiring $a_t - va_x = 0$ in the bulk.

To this end, we can check the corresponding coordinate transformations by $a_\mu dx_\mu = \tilde{a}_\mu d\tilde{x}_\mu$, where

$$\tilde{t} = t, \tilde{x} = x + vt, \tilde{y} = y \qquad (5.34)$$

Thus, $\tilde{a}_\mu = a_\nu \frac{\partial x_\nu}{\partial \tilde{x}_\mu}$

$$\tilde{a}_{\tilde{t}} = a_t - va_x = 0, \tilde{a}_{\tilde{x}} = a_x, \tilde{a}_{\tilde{y}} = a_y \qquad (5.35)$$

Thus, in the new coordinate the condition imposed by the gauge fixing is $\tilde{f}_{\tilde{x},\tilde{y}} = 0$ which means

$$\tilde{\tilde{a}}_i = \partial_i \phi \qquad (5.36)$$

Substitute this into our CS action, one arrive at

$$\begin{aligned}
S_{CS} &= -\int d^3\tilde{x} \epsilon_{ij} \tilde{a}_i \partial_{\tilde{t}} \tilde{a}_j \\
&= -\int d^3\tilde{x} (\partial_{\tilde{x}}\phi)(\partial^2_{\tilde{t},\tilde{y}}\phi) - y \leftrightarrow x \\
&= -\int d^3\tilde{x} \partial_{\tilde{y}} (\partial_{\tilde{x}}\phi)(\partial_{\tilde{t}}\phi) - y \leftrightarrow x \\
&= -\int_{y \in \partial D} d^2\tilde{x} \partial_{\tilde{t}}\phi \partial_{\tilde{x}}\phi
\end{aligned} \qquad (5.37)$$

Return back to the original coordinate, we have:

$$S = -\int d^2x \partial_t \phi \partial_x \phi - v(\partial_x \phi)^2 \qquad (5.38)$$

Which is exactly the chiral luttinger liquid Hamiltonian after defining $\rho = \frac{1}{2\pi}\partial_x \phi$ with the equation of motion $\partial_t \rho - v\partial_x \rho = 0$.

We then would like to quantize the whole chiral boson theory and find the right solution of the electron operator in terms of chiral bosonic operator. Taking the Fourier transformation of the field operator in a disk geometry(omit the zero-mode[7].):

$$\phi(\sigma, t) = \frac{1}{\sqrt{L}} \sum_{-\infty}^{\infty} \phi_q(t) \exp(iq\sigma) \tag{5.39}$$

and so does ρ_q and $q_n = \frac{2\pi n}{L}$ with the relation $\rho_q = \frac{iq_n}{2\pi}\phi_q$ and $\phi^*_q = \phi^*_{-q}, \rho^*_q = \rho^*_{-q}$. The action then becomes

$$S = -\sigma_{xy} \int dt \sum_{0}^{\infty} (iq_n \dot{\phi}_q \phi_{-q} + vq_n^2 \phi_q \phi_{-q}) \tag{5.40}$$

Which generates the commutators we use from classical theory to quantum one.

$$\begin{aligned} [\phi_q, \phi_{q'}] &= \frac{1}{\sigma_{xy} q} \delta(q + q') \\ [\rho_q, \phi_{q'}] &= \frac{i}{2\pi\sigma_{xy}} \delta(q + q') \\ [\rho_q, \rho_{q'}] &= \frac{q}{(2\pi)^2 \sigma_{xy}} \delta(q + q') \end{aligned} \tag{5.41}$$

Which is exactly an example of $U(1)$ Kac-Moody algebra. Fourier transform back to real space we will obtain our familiar equal-time commutation relations.

$$\begin{aligned} [\phi(x), \phi(x')] &= \frac{i}{2\sigma_{xy}} \text{sign}(x - x') \\ [\rho(x), \phi(x')] &= \frac{i}{2\pi\sigma_{xy}} \delta(x - x') \\ [\rho(x), \rho(x')] &= -\frac{i}{\sigma_{xy}} \partial_x \delta(x - x') \end{aligned} \tag{5.42}$$

with the Hamiltonian $H = \sigma_{xy} v \sum_0^\infty \rho_q \rho_{-q}$

The corresponding fermionic operator or charge operator is obvious when recalling previous knowledge:

$$\psi =: \exp(im\phi) : \tag{5.43}$$

which is a rewritten form of identity Eq.1.21(where $q > 0$ is annihilation operator and $q < 0$ the creation operator).

One would like to show that $[\rho(x), \psi^\dagger(x')] = \psi^\dagger(x - x')$ and $[\rho(x), \psi(x')] = -\psi(x - x')$ and thus,

$$\begin{aligned} \psi(x)\psi(x') &= \exp(-(2\pi)^2 \sigma_{xy}^2 [\phi(x), \phi(x')]) \psi(x')\psi(x) \\ &= \exp(-i2\pi^2 \sigma_{xy} \text{sign}(x - x')) \psi(x')\psi(x) \end{aligned} \tag{5.44}$$

Note that $\sigma_{xy} = \frac{\nu}{2\pi}$ we conclude that if $\nu = odd$ the operator obeys the anti-commutational relation.

One can also define the field operator of Abelian anyon with charge $\pm 1\nu$ by $\psi_{\text{anyon}} =: \exp(i\phi) :$.

[7]The rigorous treatment of the zero-mode can be found in the original paper, XG Wen, PRB, 41,18, 1990

5.4 Step 4: Bosonization: Chiral Sine-Gordan Model and Its RG Analysis

The sine-Gordan model can from my reason. Generally speaking, it's because the tunneling between different chiral sector of the bosonic field. For example, we now consider the one edge is brought proximity to another edge to form an torus. Note that only electron is permitted to tunnel between these edge. Including two edges and the tunneling effect[8]:

$$S = \int d^2x \sum_{\sigma=\pm \text{ for } R,L} \sigma \partial_t \phi_\sigma \partial_x \phi_\sigma - M_{\sigma,\sigma'} \partial_x \phi_\sigma \partial_x \phi_{\sigma'} \tag{5.45}$$

Adding the tunnel term $\Phi_\sigma^\dagger \Phi_\sigma'$ by using Eq. 4.7, and define $\phi = \phi_R + \phi_L$ we have:

$$S = \int d^2x c_1 (\nabla\phi)^2 + c_2 \cos(\beta\phi) \tag{5.46}$$

A Prove the important identity

A.1 From field theory

Our goal is to verify $\left\langle e^{iU} \right\rangle = e^{-\frac{1}{2}\left\langle U^2 \right\rangle}$ that serve as an important identity to calculate correlation function for bosonic field. Here, U is the linear combination of ϕ and θ. To achieve this, we first take the Fourier transformation of U:

$$U = \sum_j A_j \phi_j + B_j \theta_j = \frac{T}{L} \sum_k A_k \phi_{-k} + B_k \theta_{-k} \tag{A.1}$$

Where $A_k = \Sigma_j A_j e^{-i(kx_j - \omega_n \tau_j)}$ and summation of k denotes summation of (k, ω_n)

For convenience, we define $P_k = (\theta_k, \phi_k)^T, Q_k = (B_k, A_k)^T$ and obtain the L.H.S. of the identity to be proved by taking the path integral in k-space:

$$\begin{aligned} \left\langle e^{iU} \right\rangle &= \frac{1}{Z} \int \mathcal{D}P_k \mathrm{e}^{-\frac{1}{2}\frac{T}{L}\sum_k P_{-k}^T M^{-1} P_k - i\left[Q_{-k}^T P_k + P_{-k}^T Q_k\right]} \\ &= \exp\left[-\frac{1}{2}\frac{T}{L}\sum_q Q_{-k}^T M Q_k\right] \end{aligned} \tag{A.2}$$

The second line we use basic Gaussian integration, where the kernel matrix M is:

$$M = \frac{\pi}{k^2\left(k^2 + \omega_n^2\right)} \begin{pmatrix} k^2 & -ik\omega_n \\ -ik\omega_n & k^2 \end{pmatrix} \tag{A.3}$$

To take the summation in (A.2), we need to use this two Green function:

$$\begin{aligned} \mathcal{F}_1(x) &= \frac{T}{L}\sum_k \left[1 - \cos\left(kx + \omega_n \tau\right)\right] \frac{2\pi}{\omega_n^2 + k^2} \\ \mathcal{F}_2(x) &= \frac{T}{L}\sum_k e^{i(kx - \omega_n \tau)} \frac{-i2\pi\frac{\omega_n}{k}}{\omega_n^2 + k^2} \end{aligned} \tag{A.4}$$

[8]details in the original paper of X.G Wen, PRB, 1990

These two summations are exactly what we've learned from QFT. So we just write down the answer:

$$\begin{aligned}
\mathcal{F}_1(x) &= \frac{1}{2}\ln\left[\frac{1}{\pi^2\alpha^2}\left(\sinh^2(x\pi T)+\sin^2(\tau\pi T)\right)\right] \\
\mathcal{F}_2(x) &= -iArg\left[\tan\gamma_\alpha T\pi + i\tanh(x\pi T)\right] \\
\gamma_\alpha &= (u\tau + \alpha \mathrm{Sign}(\tau))
\end{aligned} \tag{A.5}$$

Where α is the cutoff of the summation.

For example, $A_{-k}A_k$ term is:

$$\begin{aligned}
&-\frac{1}{2}\frac{T}{L}\sum_{i,j,k,\omega_n} A_iA_j\cos\left(k(x_i-x_j)-\omega_n(\tau_i-\tau_j)\right)\frac{\pi}{\omega_n^2+k^2} \\
&= -\frac{1}{4}\sum_{i,j}A_iA_jF_1(x_i-x_j,\tau_i-\tau_j) + \left(\sum_i A_i\right)^2\frac{T}{2L}\underbrace{\sum_q\frac{\pi}{\omega_n^2+k^2}}_{\infty}
\end{aligned} \tag{A.6}$$

Thus, only when $\Sigma_j A_j = 0$ and $\Sigma_j B_j = 0$, the correlation function give nonzero result and becomes very simple:$e^{-\frac{1}{2}\sum\limits_{i,j}[-A_iA_j-B_iB_j]F_1+[A_iB_j+B_iA_j]F_2}$.

Experienced reader will immediately find out that the result is exactly R.H.S $e^{-\frac{1}{2}\langle U^2\rangle}$

A.2 From commutation algebra

By using the fact $e^{A+B} = e^Ae^Be^{-\frac{1}{2}[A,B]} = e^Be^Ae^{\frac{1}{2}[A,B]}$, and decomposing the operator $A = A^+ + A^-$ and $B = B^+ + B^-$ where $+$ denotes creation parts of A, B and $-$ denotes the destroy parts of A, B, we obtain:

$$\begin{aligned}
&e^Ae^B = e^{A^++A^-}e^{B^++B^-} \\
&= e^{A^+}e^{A^-}e^{B^+}e^{B^-}e^{-\frac{1}{2}\left[A^+,A^-\right]}e^{-\frac{1}{2}\left[B^+,B^-\right]} \\
&= e^{A^+}e^{B^+}e^{A^-}e^{B^-}e^{\left[A^-,B^+\right]}e^{-\frac{1}{2}\left[A^+,A^-\right]}e^{-\frac{1}{2}\left[B^+,B^-\right]}
\end{aligned} \tag{A.7}$$

Because the commutators are numbers and the cross terms from multiplication of the coeffecients of A, B, they equal the vacuum expectation values and :

$$\begin{aligned}
&e^Ae^B =: e^Ae^B : exp\left[\left\langle\left[A^-,B^+\right]-\frac{1}{2}\left[A^+,A^-\right]-\frac{1}{2}\left[B^+,B^-\right]\right\rangle\right] \\
&=: e^Ae^B : exp\frac{1}{2}\left[\left\langle 2A^-B^+-2B^+A^-+A^-A^+-A^+A^-+B^-B^+-B^+B^-\right\rangle\right] \\
&=: e^Ae^B : exp\frac{1}{2}\left[\left\langle 2A^-B^++A^-A^++B^-B^+\right\rangle\right]
\end{aligned} \tag{A.8}$$

Thus, by comparing the term from the RHS, we obtain:

$$\begin{aligned}
&\frac{1}{2}\left\langle\left(A^-+A^+\right)^2+\left(B^-+B^+\right)^2+2\left(A^-+A^+\right)\left(B^-+B^+\right)\right\rangle \\
&= exp\frac{1}{2}\left[\left\langle 2A^-B^++A^-A^++B^-B^+\right\rangle\right]
\end{aligned} \tag{A.9}$$

We arrive at:

$$e^A e^B =: e^A e^B : e^{\frac{1}{2}\left\langle (A+B)^2 \right\rangle} \tag{A.10}$$

Take the fact that, only constant terms survive in the vacuum expectation value of a normal-ordering operator, we arrive that important result:

$$\left\langle e^A e^B \right\rangle = e^{\frac{1}{2}\left\langle (A+B)^2 \right\rangle} \tag{A.11}$$

B Integral and differentiation tricks

B.1 Integral

To calculate $I(x) = \frac{1}{4\pi}\int_0^\infty dq \frac{\exp(iqx)-1}{q\exp(\alpha q)}, I(0) = 0$, we have:

$$\begin{aligned} I'(x) &= \frac{1}{4\pi}\int_0^\infty dq \frac{\exp(iqx)}{\exp(\alpha q)} = \frac{i}{4\pi}\frac{1}{\alpha - ix} \\ I &= \int_0^x I'(x')\,dx' = \frac{1}{4\pi}\ln\frac{\alpha}{\alpha - ix} \end{aligned} \tag{B.1}$$

B.2 Differentiation

To obtain the result Eq.5.16, we use the tricks $(\delta/\delta f)\mathrm{trln}G^{-1} = \mathrm{tr}(G\delta/\delta f G^{-1})$ and then :

$$\delta S/\delta\sigma = -i\mathrm{tr}(-\mathcal{G}) - \rho_0 + 2\times 1/2\times \int d^3x'\delta(x_0 - x_0')V^{-1}(\vec{x} - \vec{x}')\sigma(x') = 0 \tag{B.2}$$

which leads to:

$$\int d^3x'\delta(x_0 - x_0')V^{-1}(\vec{x} - \vec{x}')\sigma(x') = -(\rho - \rho_0) \tag{B.3}$$

The inverse integration transformation over stationary field immediately give the correct result of Eq.5.16

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