Note on 1D - Bosonization Technique

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Abstract

Physics does not depend on the representation, but your life surely does.

In this short note, we review the basic bosonization method studying the interacting 1D system [and then we review the rigorous developments and techniques. Several applications will be presented as examples of this powerful technique.](Omitted)

1 Luttinger Model: Interacting electron in 1D

We consider the nearly free electron in 1D, which is called Luttinger model. The Hamiltonian reads:

$$H = \sum_k a_k^\dagger \left(\frac{k^2}{2m} - E_F \right) a_k + \frac{1}{2L} \sum_{k,k',q\neq 0} V\left(q\right) a_{k-q}^\dagger a_{k'+q}^\dagger a_{k'} a_k \tag{1.1}$$

Where $a_k^\dagger$ denotes a spinless electron(and you can recover the spin band then, or you can simply consider a fully spin-polarized band), E_F is the chemical potential, L is the length of the system and $V(q) = \frac{4\pi}{q^2}$ denotes the interaction.

We first split the Hamiltonian into non-interacting part and the interacting term, i.e $H = H_0 + V(q)$. The most particular feature that highlights the 1D system is that only density fluctuations are essential. One can picture this by the fact that optimizing the 1D fermions, electrons can merely push each other but in higher-dimension, electron can move around to reach the optimizing state.

The 1D system's Fermi surface, only containing two isolated points $\{-k_F, k_F\}$, is also different from the higher-dimensional case. We first try to reshape the free Hamiltonian, in order to obtain some insights about the problem.

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1.1 Expanding the free Hamiltonian

It's natural to expand the system near Fermi surface as the collective exicitations are merely related to the physics that happens near the Fermi surface.

By using the Taylor expansion:

$$\frac{k^2}{2m} = \frac{k_F^2}{2m} + \sigma v_F (k - k_F) + O\left(k^2\right) \tag{1.2}$$

Where $\sigma = \pm$ denote the right-moving fermion($+$) and the left-moving fermion($-$).

The Hamiltonian H_0 can be approximated as:

$$H = \sum_{k,\sigma} v_F (\sigma k - k) a_{\sigma,k}^{\dagger} a_{\sigma,k} \tag{1.3}$$

Which can be immediately recognized as a Dirac Hamiltonian. It should be always remembered that the summation is taken when the linear approximation is validate, i.e, $|k - k_F| < \Lambda$, a cut-off is added to the summation. As the basic excitation of this model is particle-hole excitation, one wants to test if the particle-hole quasi-excitation have well-defined momentum and other quantum number, which can be achieved by write down the spectrum of particle-hole:

$$E_{R,k}(q) = E_R(k+q) - E_R(k) = v_F(k+q) - v_F k = v_F q \tag{1.4}$$

Thus, the linear system have well-defined particle-hole excitations. So it's then natural to using the corresponding operator $\Sigma_k a_{\sigma,k+q}^{\dagger} a_{\sigma,k}$ to rewrite our Hamiltonian. This is a very important point, from which we develop a whole method named 'bosonization'. One can also check directly that this operator is the Fourier transform of the density operator:

$$\begin{aligned} \rho_q &= \frac{1}{2\pi} \int dx e^{iqx} \rho(x) = \frac{1}{2\pi} \int dx e^{iqx} a_x^{\dagger} a_x \\ &= \left(\frac{1}{2\pi}\right)^3 \sum_{k,k'} \int dx e^{iqx} e^{ikx} a_k^{\dagger} e^{-ik'x} a_{k'} = \sum_k a_{k+q}^{\dagger} a_k \end{aligned} \tag{1.5}$$

And naturally $\rho_q^{\dagger} = \rho_{-q}$ as the density operator is Hermitian.

1.2 Calculating the of the density operator

Before we go further to demonstrate the idea, we should be cautious to avoid infinities when treating the Fourier transformation of (i.e, a linear combination of) the density operator. Because in Dirac-like Hamiltonian, it is permitted that infinite occupation below the Fermi energy. To this end, we introduces so-called normal ordering tricks:

In a normal ordered product, the destruction operators are put on the right and creation operators on the left. For two operator A, B that are linear combinations of creation and destruction operator, normal ordering operation is equivalent to the original operator subtracting the average value of the operator in the vacuum, i.e:

$$: AB := AB - \langle 0| AB |0\rangle \tag{1.6}$$

Then our goal is to figure out the commutation relation of this density operator, and take the linear superposition of them to recover the Hamiltonian Eq(3). The fronter procedure can be accomplished by writing down:

$$\begin{aligned}[\rho_{\sigma,q},\rho_{\sigma,-q'}] &= \sum_{k1,k2}\left[a^\dagger_{\sigma,k1+q}a_{\sigma,k1},a^\dagger_{\sigma,k2-q'}a_{\sigma,k2}\right] \\ &= \sum_{k1k2} a^\dagger_{k1+q}a_{k2}\delta_{k1,k2-q'} - a^\dagger_{k2-q'}a_{k1}\delta_{k2,k1+q} \\ &= \sum_{k2} a^\dagger_{\sigma,k2+p-p'}a_{\sigma,k2} - a^\dagger_{\sigma,k2-p'}a_{\sigma,k2-p}\end{aligned} \tag{1.7}$$

In the second line we omit the right-left index for simplicity. We may naively change the index of summation and conclude that the result is zero, but which turns out to be wrong:

$$\sum_{k2} a^\dagger_{\sigma,k2+p-p'}a_{\sigma,k2} - \sum_{k2-p\to k2} a^\dagger_{\sigma,k2+p-p'}a_{k2} \overset{?}{=} 0$$

The reason is that the bare density operators(Fourier-transformed) contain infinity number of occupied states and the equation becomes $\infty-\infty$ indefinite.(as an example, when k involves infinite modes, that $\sum_k b_k b_k^\dagger \overset{!}{=} \sum_k 1 + b_k^\dagger b_k = \infty + \sum_k b_k^\dagger b_k$) So we try to use the normal ordering tricks to evaluate this quantity because the matrix element of normal ordering operator is always finite. As a matter of fact:

$$\begin{aligned}[\rho_{\sigma,q'},\rho_{\sigma,-q'}] &= \sum_{k2} :a^\dagger_{\sigma,k2+q-q'}a_{\sigma,k2}: - :a^\dagger_{\sigma,k2-q'}a_{\sigma,k2-q}: \\ &+ \sum_{k2} \langle 0|\, a^\dagger_{\sigma,k2+q-q'}a_{\sigma,k2}\,|0\rangle - \langle 0|\, a^\dagger_{\sigma,k2-q'}a_{\sigma,k2-q}\,|0\rangle\end{aligned} \tag{1.8}$$

Then, the first subtraction can be safely evaluated as zero. Making use of the fact that $\langle 0|\, a^\dagger_{\sigma k}a_{\sigma k'}\,|0\rangle = \delta_{k,k'}$ we then get:

$$[\rho_{\sigma,q'},\rho_{\sigma,-q'}] = 0 + \delta_{q,q'}\sum_{k2}\langle 0|n_{\sigma,k2} - n_{\sigma,k2-q}\,|0\rangle \tag{1.9}$$

One may then naively to conclude that the second term equals zero because at a first glance

$$\sum_{k2}\langle 0|n_{\sigma,k2}\,|0\rangle \overset{?}{=} \sum_{k2}\langle 0|n_{\sigma,k2-q}\,|0\rangle$$

But this is not true because we actually have a cut-off in momentum! Since the shift $k\to k-p$ changes the cut-off, the result is exactly the number of state in the interval $[k,k+q]$. Due to the well defined properties of the density in 1-D system, the result is $\frac{|q|}{2\pi/L}$ and independent of the cut-off properties of Λ. We arrive at the conclusion:

$$[\rho_{\sigma,q},\rho_{\sigma',-q'}] = -\delta_{\sigma,\sigma'}\delta_{q,q'}\frac{\sigma qL}{2\pi} \tag{1.10}$$

The result is remarkable because of the bosonic properties of density operator arising from the large number of occupied states we consider. Thus we are able to recover the Hamiltonian we care about with bosonic density fluctuations.

1.3 Bosonization of Luttinger Model

We now construct the bosonic operator from the density operator. The method is direct to some extents, but it can also provide some insights.

Note that, in the RHS of the Eq 1.10, the commutator is momentum dependent. The natural way to remove this dependency is to partition the coefficients to each density operators, i,e , $\rho_{\sigma,q} \to \frac{\rho_{\sigma,q}}{(\frac{L|q|}{2\pi})^{1/2}}$.

Due to some sign considerations, one can finally (after some nontrivial trials) write down the bosonization solution for this problem:

$$\begin{aligned} b_q &= n_q \rho_{+,q}, b_q^\dagger = n_q \rho_{+,-q} \\ b_{-q} &= n_q \rho_{-,-q}, b_{-q}^\dagger = n_q \rho_{-,q} \end{aligned} \tag{1.11}$$

Where $n_q = (\frac{2\pi}{L|q|})^{1/2}$ and $q > 0$.

Compactly and equivalently: $b_q^\dagger = n_q \sum_\sigma \Theta(\sigma q)\, \rho_{\sigma,-q}$, where Θ is the Heaviside theta function.

Having constructed the boson operator$(b, b^\dagger)$, we are now to find a representation. Before diving into the details, we can observe that a boson operator consists two fermion operator If we can use two boson operators to represent free Hamiltonian, it means that original two fermion operator should be mapped to quadratic fermion operators, e.g ,$a^\dagger a a^\dagger a = a^\dagger \left[\delta - a^\dagger a\right] a = \delta a^\dagger a - \left(a^\dagger a^\dagger a a\right)$(neglect the subscripts). This also indicates that, if it's possible to bosonize the free Hamiltonian, the interacting part is also quadratic and supposed to be bosonized.

The Hamiltonian of a system can be constructed in many forms but the evolution of the observable remains intact. As the Heisenberg equation indicate, the only important element is the commutation relation between the complete operator and the Hamiltonian. For example, we try to calculate the anti-commutation rules of $b_q (q > 0)$ and H_0:

$$\begin{aligned} [b_q, H_0] &= n_q \sum_\sigma \left[\rho_{+,p}, v_F\,(\sigma k - k_F)\, a_{\sigma,k}^\dagger a_{\sigma,k}\right] \\ &= n_q \sum_{k,k_q} v_F\,(k - k_F) \left(a_{+,k1-q}^\dagger a_{+,k} \delta_{k1-k} - a_{+,k}^\dagger a_{+,k1} \delta_{k1-q-k}\right) \\ &= n_q \sum_k v_F q c_{+,k-q}^\dagger c_{+,k} = v_F q b_q \end{aligned} \tag{1.12}$$

and similar relation holds when $q < 0$. We can ultimately write down the transformed Hamiltonian:

$$H_0 \simeq const + \sum_{p \neq 0} v_F\, |p|\, b_p^\dagger b_p \tag{1.13}$$

And rewriting the interaction part is obvious:

$$\begin{aligned} V &= \frac{1}{2\pi}\sum_{q>0} q\left(g_4 b_q b_q^\dagger + g_4 b_{-q}^\dagger b_{-q} + g_2 b_q b_{-q} + g_2 b_{-q}^\dagger b_q\right) \\ &= \frac{1}{2\pi}\sum_{q>0} q\begin{pmatrix} b_q & b_{-q}^\dagger \end{pmatrix} K_{ee}\begin{pmatrix} b_q^\dagger \\ b_{-q}\end{pmatrix} \\ K_{ee} &= \begin{pmatrix} g_4 & g_2 \\ g_2 & g_4 \end{pmatrix} \end{aligned} \tag{1.14}$$

The interaction part is also quadric but contains non-particle number-conserving contributions. Together with free Hamiltonian, we can use Bogoliubov transformation to diagonalize the Hamiltonian which is a trivial task.

To diagonalize the Hamiltonian:

$$H = \sum_{q>0} q\Psi_q^\dagger K \Psi_q, \Psi_q = \begin{pmatrix} b_q^\dagger \\ b_{-q}\end{pmatrix} \tag{1.15}$$

We need to find a transformation to diagonalize the kernel matrix and also keep the generalized commutation relation intact. We find that:

$$C_{ij} = \left[\Psi_{q,i}, \Psi_{q,j}^\dagger\right] = \begin{pmatrix} -1 & \\ & 1 \end{pmatrix}_{ij} = (-\sigma_3)_{ij}$$

So the transformation matrix U should satisfy those condition:

$$\begin{aligned} H &= \sum_{q>0} q\Psi_q'^\dagger U^\dagger K U \Psi'_q \\ U^\dagger K U &= \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \\ U^\dagger \sigma_3 U &= \sigma_3 \end{aligned} \tag{1.16}$$

To this end, we find $\underbrace{\sigma_3 U^\dagger \sigma_3}_{U^{-1}} \sigma_3 K U = \sigma_3 K'$ which means$\sigma_3 K'$ contains the eigenvalues $\pm u$ on its diagonal. and $\sigma_3 K' = Diag(u, -u)$ means $K' = Diag(u, u)$. The eigenvalue of $\sigma_3 K$ can be readily computed as $u = \frac{\sqrt{(g_4 + 2\pi v_F)^2 - g_2^2}}{2\pi}$, we then arrive at the final result:

$$H = u\sum_{q>0} q\Psi_q^\dagger \Psi'_q = u\sum_q |q|\, b_q^\dagger b_q \tag{1.17}$$

The observables, particularly some thermodynamics, can be extracted from this Hamiltonian. The easiest calculation among them is the heat capacity:

$$\begin{aligned} C_V &= \frac{dE}{dT} = \frac{d}{dT}\sum_{p\neq 0} \varepsilon(p)\, f_B(\varepsilon(p)) \\ &= \beta^2 \sum_{p\neq 0} \varepsilon(p)^2 \frac{e^{\beta\varepsilon}}{\left(e^{\beta\varepsilon} - 1\right)^2} = \frac{T}{u}\left(\frac{L\pi}{3}\right) \end{aligned} \tag{1.18}$$

One can compare this result with free fermion gas case: $C_V = \frac{T}{v_F}(\frac{L\pi}{3})$

This kind of quasi excitation as the Bosonic operators describing the density fluctuations is termed charge density wave with the velocity u. In the non-interacting limit, where $g_2 = g_4 = 0$, the waves have a velocity v_F while $g_2 = 0, g_4 \neq 0$ will accelerate the wave. Intuitively, this can be interpreteda as the nonzero g_4 is the fictitious interaction. But when $g_2 \neq 0$, the Coulomb interaction make it difficult for particles with opposite velocity move close to each other.

1.4 Further consideration

We are now at a stage that the Hamiltonian is quadratic combination of the boson p mode operator. To develop a corresponding field theory, we have a standard routine, which is exactly the inverse routine that we learned in standard QFT or Quantum Optics.

We then consider the transformed field operator using the same technique we developed before. The fermionic field operator is defined as the Fourier transformation of the mode annihilation operator:

$$a_\sigma(x) = \frac{1}{\sqrt{2\pi L}} \sum_k e^{ikx} a_{\sigma,k} \tag{1.19}$$

As the same spirit that the commutation relation can define the observables, we have :

$$\begin{aligned} \left[\rho_\sigma(q), a_\sigma(x)\right] &= \frac{1}{\sqrt{2\pi L}} \sum_{k,k_1} e^{ik1x} \left[c^\dagger_{\sigma,k+q} c_{r,k}, c_{r,k1}\right] \\ &= -e^{ipx} a_\sigma(x) \end{aligned} \tag{1.20}$$

So, we can construct the field operator as $a_\sigma(x) \overset{!}{\simeq} e^{\sum_q e^{iqx} \rho_\sigma(q) \left(\frac{2\pi r}{qL}\right)}$. But it's not true, because the density operator never create or destroy any fermion but the field operator destroy a fermion in x. One wants to generate this as :

$$a_\sigma(x) = U_\sigma e^{\sum_q e^{iqx} \rho_\sigma(q) \left(\frac{2\pi r}{qL}\right)} \tag{1.21}$$

Where the U_σ should destroy a fermion of state σ and commute with the boson operator. There operators are known as 'Klein factor'. Using the conditions above, the dependence of the Klein factor on the boson operator can be derived. But in the first example, it's too tedious to do so, and few physics can be extracted by Klein factor, though this provides an exact and full transformation from fermionic representation to a bosonic one.

2 Bosonization with a field theory

We now want to develop a bosonic field theory about the Luttinger model. Put it in you heart that we even know the right answer to the theory because of the universality of the wave mechanism :

$$\mathcal{H} = \frac{\pi}{v_F} \Pi^2 + \frac{v_F}{2\pi} \partial_x \theta^2 \tag{2.1}$$

Thus, all we need to do is to reshape and map our original free system to this bosonic field theory. In other words, we want to recover the field Langragian density and Hamiltonian from the mode operator representation. To obtain an effective long range theory of the 1D interacting system, we have two strategies. The first one is to explictly construct the field theory from the mode operator and the action of the operator on Fock state. Another more phenomenological approach is based on consideration of the consistency and the symmetry acquirement. Although this method lacks quantitative evaluation of the results, we can gain some intuitions and even immediately obtain the right theory and finally make it rigorous.

2.1 Euclidean Fermionic Field Theory

We use the language of Euclidean fermionic field theory to represent this system:

$$\begin{aligned} S_0\left[\psi^\dagger,\psi\right] &= \sum_\sigma \int dx d\tau \psi_\sigma^\dagger \left(-i\sigma v_F \partial_x + \partial_\tau\right)\psi_\sigma \\ S_{\text{int}}\left[\psi^\dagger,\psi\right] &= \frac{1}{2}\sum_\sigma \int dx d\tau \left(g_2 \rho_\sigma \rho_{\sigma'} + g_4 \rho_s \rho_s\right) \end{aligned} \tag{2.2}$$

As before, $\sigma = \pm$ and $+$ for right-moving fermions while $-$ for left-moving ones.

We will first show the phenomenological method. As we see, the 1D interacting system can hold density wave excitation, so the fermionic part should change the density of the system. As $N = \frac{L}{2\pi} 2k_F \to n_0 = \frac{k_F}{\pi}$, the first part of the bosonic operator is supposed to be $\frac{k_F}{\pi} + \rho\left(x\right)$. So we can write down the transformation:

$$b\left(x\right) = \left(\frac{k_F}{\pi} + \rho\left(x\right)\right)^{1/2} e^{i\phi(x)} \tag{2.3}$$

It can be easily shown that the transformation $(b, b^\dagger) \to (\rho, \phi)$ is canonical. For simplicity, we recover the total density of the system $\rho \to \rho + \frac{k_F}{\pi}$, we thus have:

$$\begin{aligned} 1 &= \left[b, b^\dagger\right] = \left[\rho^{1/2} e^{i\lambda\phi}, e^{-i\lambda\phi}\rho^{1/2}\right] \\ &= \rho - e^{-i\phi}\rho e^{i\phi} = \rho - e^{-i[\phi,]}\rho = \rho - \rho + i\left[\phi,\rho\right] - \frac{1}{2}\left[\phi,\left[\phi,\rho\right]\right] + ... \end{aligned} \tag{2.4}$$

The equality thus require $[\rho, \phi] = i$ whose complete and continuous version is:

$$[\rho(x), \phi(x')] = i\delta(x - x') \tag{2.5}$$

Based on our experience of Jordan-Wigner transformation, fermionic operator afford a representation of bosons: $a^\dagger\left(x\right) = e^{i\pi\int_{x'<x} dx' n\left(x'\right)} b^\dagger\left(x\right), n\left(x\right) = b^\dagger\left(x\right) b\left(x\right)$ where $b(x)$ create a boson located at x and $\exp(i\pi \int_{x'<x} dx' n\left(x'\right))$ implements the fermionic statistics (What is fermion? Fermion is a hardcore boson in the string. This exponent operator serves as a string.) Thus, one can apply basic Jordan-Wigner transformation to construct fermionic representation. However, after simple trial you will find the solution is not corresponding to what we want, though right but useless. In this case, we can also implement this routine by define $\theta(x) = \pi \int_{-\infty}^{x} dx' \rho(x')$. Exponent of this will serve as a string operator. So what it the annihilator or creation operator? Note that the momentum conjugated to ρ is ϕ and the corresponding translation operator is

$e^{i\phi(x)}$ creating a unity charge located at x. So the construction is naturally:

$$a^{\dagger}(x) = A\sum_{\sigma} e^{i\sigma\theta(x)/\pi} e^{i\phi(x)} \tag{2.6}$$

where A is introduced to regularize the RHS. And dividing it into the right-moving part and left-moving part:

$$a^{\dagger}_{\sigma}(x) = Ae^{i\sigma\theta(x)/\pi} e^{i\phi(x)} \tag{2.7}$$

After shifting $\rho \to \rho + \frac{k_F}{\pi}$, and corresponding $\theta \to \theta + k_F x$, we get the final answer:

$$a^{\dagger}(x) = A\sum_{\sigma} e^{i\sigma\pi^{-1}[\theta(x)+k_F x]} e^{i\phi(x)} \tag{2.8}$$

And so does $a^{\dagger}_{\sigma}$ The commutation relation of our two main fields is

$$[\phi(x), \theta(x')] = i\pi\Theta(x'-x) \tag{2.9}$$

2.2 Bosonic Field in terms of bosonic mode

We've learned in basic quantum field theory that bosonic field operator can be represented by bosonic mode operator as:

$$\begin{aligned}\phi(x) &= \int \frac{dp}{2\pi\sqrt{2|p|}}\left[\phi_p e^{ipx} + \phi^{\dagger}_p e^{-ipx}\right] e^{-\frac{1}{2}\alpha|p|} \\ \Pi(x) &= \int \frac{dp}{2\pi\sqrt{2|p|}}\left[-i\phi_p e^{ipx} + i\phi^{\dagger}_p e^{-ipx}\right] e^{-\frac{1}{2}\alpha|p|}\end{aligned} \tag{2.10}$$

It's supposed to be emphasized that in bosonization theory, we have actually the same identity:

$$\begin{aligned}\phi(x) &= \lim_{L\to\infty}\sum_p \left(\frac{1}{2\pi L|p|}\right)^{1/2} Sign(p)\, e^{-\alpha|p|/2-ipx}\left(b^{\dagger}_p + b_{-p}\right) \\ \theta(x) &= \lim_{L\to\infty}\sum_p \left(\frac{1}{2\pi L|p|}\right)^{1/2} e^{-\alpha|p|/2-ipx}\left(b^{\dagger}_p - b_{-p}\right)\end{aligned} \tag{2.11}$$

One can easily check this by previous knowledge.

2.3 Construction of Phenomenological Field Theory

We now construct the action of this field theory. First, inspired by Eq. 2.7, we know that the axial and vectorial symmetry are required: shifting $(\phi, \theta) \to (\phi + \phi_v, \theta + \phi_a)$. For constant $\phi_{a,v}$, the action is intact. Thus the non-derivative terms are excluded.

Next, the density distortion $\rho = \partial_x\theta/\pi$ costs certain energy. Due to the screening effect, the coulomb interaction should be short range i.e local. Up to second order in derivatives, the action contain term like $(\partial_x\theta)^2$. To ensure rotational invariance in the $1+1$-D space-time, the total action is given by $S_0[\theta] = \frac{c}{2}\int dx d\tau \left((\partial_\tau\theta)^2 + (\partial_x\theta)^2\right)$

Now our free theory have two undetermined parameter regularization coefficient A and the coupling constant c. As the free theory is integrable in fermion representation, we can compare the observables forecast by those two theories (e.g capacity). For example, we will calculate $C(x,\tau) = \left\langle \left(\psi_+^\dagger \psi_-\right)(x,\tau)\left(\psi_-^\dagger \psi_+\right)(0,0)\right\rangle$ for bosonic theory. Corresponding quantity in fermionic theory is $C(x,\tau) = A^4\left\langle e^{2i\theta(x,\tau)} e^{-2i\theta(0,0)}\right\rangle_\theta$. By using Matsubara summation, we have :

$$\begin{aligned} G_\pm &= -\frac{1}{\partial_{\tau'} \mp i\partial_{x'}}\bigg|_{(x,\tau;0,0)} \\ &= -\frac{T}{L}\sum_{p,\omega_n} \frac{1}{-i\omega_n \mp p} e^{-ipx - i\omega_n\tau} \\ &\simeq \frac{1}{2\pi}\frac{1}{\pm ix - \tau} \end{aligned} \tag{2.12}$$

We've used an integral to approximate the frequency sum and integrate over momentum. By wick's theorem, the correlation function is:

$$C = G_+ G_- = \frac{1}{4\pi^2}\frac{1}{x^2+\tau^2}$$

For bosonic system. With the interaction $S[\theta] = \frac{L}{2cT}\Sigma_{q,n}(q^2+\omega_n^2)|\theta_{q,n}|$ also by Matsubara summation, we have the green function for θ field:

$$\begin{aligned} K(x,\tau) &= \left\langle \theta(x,\tau)\,\theta(0,0) - \theta^2(0,0)\right\rangle \\ &= \frac{cT}{L}\sum_{p,n}\frac{e^{ipx} + i\omega_n\tau}{\omega^2+p^2} - 1 = \frac{cT}{2}\sum_n \frac{e^{-|\omega_n|x + i\omega_n\tau} - 1}{|\omega_n|} \\ &= \frac{c}{4\pi}\int_0^{a^{-1}} d\omega \frac{e^{-\omega(x-i\tau)} - 1}{\omega} + \frac{e^{-\omega(x+i\tau)} - 1}{\omega} \\ &\simeq -\frac{c}{4\pi}\ln\frac{x^2+\tau^2}{a^2} \end{aligned} \tag{2.13}$$

Where we add a cutoff a^{-1} and use stationary phase approxiamtion in large x and τ.

By using the fact $\left\langle e^{iU}\right\rangle = e^{-\frac{1}{2}\langle U^2\rangle}$ Where U is the linear combination of ϕ and θ(See the prove in appendix), we have:

$$\begin{aligned} &\left\langle e^{iU}\right\rangle = e^{-\frac{1}{2}\langle U^2\rangle} \\ &C(x,\tau) = A^4\left\langle e^{2i(\theta(x,\tau)-\theta(0,0))}\right\rangle \\ &= A^4 e^{-2\left\langle(\theta(x,\tau)-\theta(0,0))^2\right\rangle} \\ &= A^4 e^{4K(x,\tau)} \\ &= A^4\left[\frac{a^2}{x^2+\tau^2}\right]^{\frac{c}{\pi}} \end{aligned} \tag{2.14}$$

Thus, comparing 2.12 and 2.14 we have $A = \frac{1}{\sqrt{2\pi a}}$ and $c = \pi$ where a^{-1} is the cutoff.

We finally obtain the Lagrangian of the bosonic version:

$$\begin{aligned}\mathcal{L} &= \frac{1}{2\pi}\left[(\partial_x\theta)^2 + (\partial_\tau\theta)^2\right] \\ \pi_\theta &= \partial_{\partial_\tau\theta}\mathcal{L} = \frac{\partial_\tau\theta}{\pi}\end{aligned} \tag{2.15}$$

Thus we have the commutation relation $[\theta(x), \pi(x')] = -i\delta(x - x')$. Comparing this with $[\phi(x), \theta(x')] = i\pi\Theta(x' - x)$, we thus have $\pi_\theta = \frac{\partial_x\phi}{\pi}$. Transforming the action from canonical variable to fundamental fields, we have:

$$\begin{aligned}S[\theta, \pi_\theta] &= \frac{1}{2}\int dxd\tau\left(\frac{1}{\pi}(\partial_x\theta)^2 + \pi\pi_\theta^2 + 2i\pi_\theta\partial_\tau\theta\right) \\ S[\theta, \phi] &= \frac{1}{2\pi}\int dxd\tau\left((\partial_x\theta)^2 + (\partial_x\phi)^2 + 2i\partial_\tau\theta\partial_x\phi\right)\end{aligned} \tag{2.16}$$

The corresponding conserved quantity of translation symmetry is

$$\begin{aligned}\rho_{total} &= \rho_+ + \rho_- = \frac{\partial_x\theta}{\pi} \\ j &= \rho_+ - \rho_- = -\frac{\partial_x\phi}{\pi}\end{aligned} \tag{2.17}$$

Thus we have the density operator in terms of field operators:

$$\rho_\pm = \frac{1}{2\pi}\left[\partial_x\theta \mp \partial_x\phi\right] \tag{2.18}$$

2.4 Interacting System

By inserting Eq 2.18 into the interaction vertex Eq (2.2), we have:

$$S_{\text{int}} = \frac{1}{4\pi^2}\int dxd\tau\left[(g_2 + g_4)(\partial_x\theta)^2 + (g_4 - g_2)(\partial_x\phi)^2\right] \tag{2.19}$$

Together with the free part, we arrive at the final expression:

$$S = \frac{1}{4\pi^2}\int dxd\tau\left[g^{-1}v(\partial_x\theta)^2 + gv(\partial_x\phi)^2 + 2i\partial_\tau\theta\partial_x\phi\right] \tag{2.20}$$

Where

$$\begin{aligned}v &= \left[\left(v_F + \frac{g_4}{2\pi}\right)^2 - \left(\frac{g_2}{2\pi}\right)^2\right]^{1/2} \\ g &= \left[\frac{v_F + \frac{g_4}{2\pi} - \frac{g_2}{2\pi}}{v_F + \frac{g_4}{2\pi} + \frac{g_2}{2\pi}}\right]^{1/2}\end{aligned} \tag{2.21}$$

The velocity we introduce here is exactly the effective velocity we derived with mode operator and diagonalization in (1.17).

2.5 Functional Bosonization

In this section, we use basic functional method to implement bosonization. In previous sections, we bosonize the theory by transforming the free part of the fermionic Hamiltonian. Now we approach the result with the idea of decoupling the quadric field by auxiliary fields.

We first rewrite the interaction part:

$$S_{\text{int}} = \frac{1}{2}\int d\tau dx \rho_i g_{ij} \rho_j \tag{2.22}$$

Where Einstein convention is assumed and $g_{11} = g_{22} = g_2, g_{12} = g_{21} = g_4$

By invoking the identity: $\exp\left[-\rho_m V_{mn} \rho_n\right] = \int \mathcal{D}\phi \left[-\frac{1}{4}\phi_m V_{mn}^{-1} \phi_n - i\phi_m \rho_m\right]$, we have:

$$S\left[\phi, \bar{\psi}, \psi\right] = \int dx d\tau \left[\frac{1}{2}\phi_i g_{ij}^{-1} \phi_j - i\phi_i \rho_i + \psi_i^\dagger (-i\sigma_3 \partial_x + \sigma_0 \partial_\tau)_{ij} \psi_j\right] \tag{2.23}$$

A Prove the important identity

Our goal is to verify $\left\langle e^{iU}\right\rangle = e^{-\frac{1}{2}\left\langle U^2\right\rangle}$. To achieve this, we first take the Fourier transformation of U:

$$U = \sum_j A_j \phi_j + B_j \theta_j = \frac{T}{L}\sum_k A_k \phi_{-k} + B_k \theta_{-k} \tag{A.1}$$

Where $A_k = \Sigma_j A_j e^{-i(kx_j - \omega_n \tau_j)}$ and summation of k denotes summation of (k, ω_n)

For convenience, we define $P_k = \left(\theta_k, \phi_k\right)^T, Q_k = \left(B_k, A_k\right)^T$ and obtain the L.H.S. of the identity to be proved by taking the path integral in k-space:

$$\begin{aligned} \left\langle e^{iU}\right\rangle &= \frac{1}{Z}\int \mathcal{D}P_k \mathrm{e}^{-\frac{1}{2}\frac{T}{L}\sum\limits_k P_{-k}^T M^{-1} P_k - i\left[Q_{-k}^T P_k + P_{-k}^T Q_k\right]} \\ &= \exp\left[-\frac{1}{2}\frac{T}{L}\sum_q Q_{-k}^T M Q_k\right] \end{aligned} \tag{A.2}$$

The second line we use basic Gaussian integration, where the kernel matrix M is:

$$M = \frac{\pi}{k^2\left(k^2 + \omega_n^2\right)} \begin{pmatrix} k^2 & -ik\omega_n \\ -ik\omega_n & k^2 \end{pmatrix} \tag{A.3}$$

To take the summation in (A.2), we need to use this two Green function:

$$\begin{aligned} \mathcal{F}_1(x) &= \frac{T}{L}\sum_k \left[1 - \cos\left(kx + \omega_n \tau\right)\right] \frac{2\pi}{\omega_n^2 + k^2} \\ \mathcal{F}_2(x) &= \frac{T}{L}\sum_k e^{i(kx - \omega_n \tau)} \frac{-i2\pi \frac{\omega_n}{k}}{\omega_n^2 + k^2} \end{aligned} \tag{A.4}$$

These two summations are exactly what we've learned from QFT. So we just write down the answer:

$$\begin{aligned}
&\mathcal{F}_1(x) = \frac{1}{2}\ln\left[\frac{1}{\pi^2\alpha^2}\left(\sinh^2(x\pi T) + \sin^2(\tau\pi T)\right)\right] \\
&\mathcal{F}_2(x) = -iArg\left[\tan\gamma_\alpha T\pi + i\tanh(x\pi T)\right] \\
&\gamma_\alpha = (u\tau + \alpha Sign(\tau))
\end{aligned} \tag{A.5}$$

Where α is the cutoff of the summation.

For example, $A_{-k}A_k$ term is:

$$\begin{aligned}
&-\frac{1}{2}\frac{T}{L}\sum_{i,j,k,\omega_n} A_iA_j\cos\left(k(x_i - x_j) - \omega_n(\tau_i - \tau_j)\right)\frac{\pi}{\omega_n^2 + k^2} \\
&= -\frac{1}{4}\sum_{i,j} A_iA_jF_1(x_i - x_j, \tau_i - \tau_j) + \left(\sum_i A_i\right)^2 \frac{T}{2L}\underbrace{\sum_q \frac{\pi}{\omega_n^2 + k^2}}_{\infty}
\end{aligned} \tag{A.6}$$

Thus, only when $\Sigma_j A_j = 0$ and $\Sigma_j B_j = 0$, the correlation function give nonzero result and becomes very simple:$e^{-\frac{1}{2}\sum\limits_{i,j}[-A_iA_j - B_iB_j]F_1 + [A_iB_j + B_iA_j]F_2}$.

Experienced reader will immediately find out that the result is exactly R.H.S $e^{-\frac{1}{2}\langle U^2\rangle}$